ANALYSIS OF AN AIR TRANSPORTATION SYSTEM

By

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A DISSERTATION PRESENTED TO THE GRADUATE SCHOOL
OF THE UNIVERSITY OF FLORIDA IN
PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

1986

ACKNOWLEDGMENTS

Praise be to GOD, the Cherisher and Sustainer of the Worlds, Most Gracious, Most Merciful, Master of the Day of Judgment. May GOD's blessings and prayers be upon his messenger Mohammed.

I am very much obliged to my advisor, Professor Thom J. Hodgson, for his encouragement, guidance, beneficial suggestions and, above all, his patience and understanding. My sincere thanks go to him.

I would also like to thank the other members of my Ph.D. dissertation committee: Dr. Louis Martin-Vega, Dr. Boghos Sivazlian and Dr. Antal Majthay.

The help and support of the University of Petroleum and Minerals of Dhahran, Saudi Arabia, are most appreciated.

My special thanks go to my wife, who stood by my side and accompanied me through the ups and downs that come into the life of many Ph.D. students.

I would also like to thank my dear friend, Mr. Abdullah Al-Mahdi, for his timely and invaluable help in preparing for the final dissertation's defense.

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December 1986

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This dissertation addresses the problem of transporting personnel, equipment and supplies over a long distance using military transport aircraft (A/C), of which one or more may need to use aerial refueling. It is desired to assist the decision maker in answering the following questions: (1) How many A/C are needed? (2) How should the total load be divided among all the used A/C? (3) What should the coordinates (location) of each refueling point be? (4) What should the amount of initial fuel be in order to minimize the total fuel consumption used to complete the move?

The aim of this dissertation is to analyze the general A/C mid-air refueling problem, and to use the insights obtained from the analysis to characterize and find the optimal solution. It is found that even though a mathematical model for the general problem can be formulated, it would provide little insight and would be too complicated to solve. Instead, the analysis approach used in this dissertation consists of breaking the general problem down into smaller and simpler

(sub)problems that are addressed in order of increasing complexity. First, the various relationships between fuel consumption, distance traveled, cargo weight, and initial fuel for a typical jet A/C are derived. Next, formulations and solution procedures of the first five (sub)problems are presented. These include both the problem of many A/C with one refueling per A/C required, and the problem of one A/C with two refuelings required. The spherical convexity of these problems is shown.

CHAPTER I

INTRODUCTION

Problem Description and Model

Consider the following scenarios: A decision is made to transport military personnel, equipment and supplies from an air force base in the United States to Cairo, Egypt to aid in the deployment of a squadron of F-15 fighters to Egypt. A similar decision could have been made to provide needed military supplies as in the case of the airlift to Israel during the 1973 Arab-Israeli War. A third case might be a Rapid Deployment Force where military personnel and light equipment are transferred to a sensitive area to intervene in case of an emergency, e.g., the invasion of Grenada in 1983.

All of the above situations are similar in the following sense. Military transport aircraft (A/C) are used to effect the move. The equipment and supplies are either packaged on pallets or consist of rolling equipment large enough to be transported as self-contained units. Equipment and men are loaded into the transport A/C, taking into consideration various physical restrictions. These restrictions include factors such as type and availability of A/C, as well as loading considerations such as maximum weight limit, type and size of equipment, and many others.

Given a deployment situation, another problem arises from the fact that the distance between the origin base and the desired destination is sometimes so great that it is necessary for the transport A/C to refuel

en route. Many times, there are air bases along the way where the transport A/C could land and refuel. However, refueling may be performed more efficiently in mid-air using special A/C called "tankers," designed for this purpose. The tanker flies to a point where it meets the transport A/C and supplies it with fuel while both are in the air. The objective is to complete the move so as to minimize the total fuel consumption of all the A/C involved.

In analyzing such a problem, the following decision variables need to be specified:

1. number of transport A/C of each type that are needed
2. weight of the load going into each A/C
3. initial fuel each A/C should carry
4. transport A/C route
5. tanker A/C route

Specifying 4 and 5 is equivalent to determining the location of all refueling points. Figure 1-1 and 1-2 show some examples of possible routes.

Figure 1-1 illustrates the case of two tanker bases. The transport flies to the first refueling point where it meets a tanker coming from the first base and refuels in mid-air. After aerial refueling is finished, the tanker flies to its base while the transport flies to the second refueling point where it meets another tanker from the second base. Then, after refueling, the transport heads directly for the desired destination. Figure 1-2 depicts a case where there is only one tanker base, but aerial refueling is done twice using tankers originating from this base.

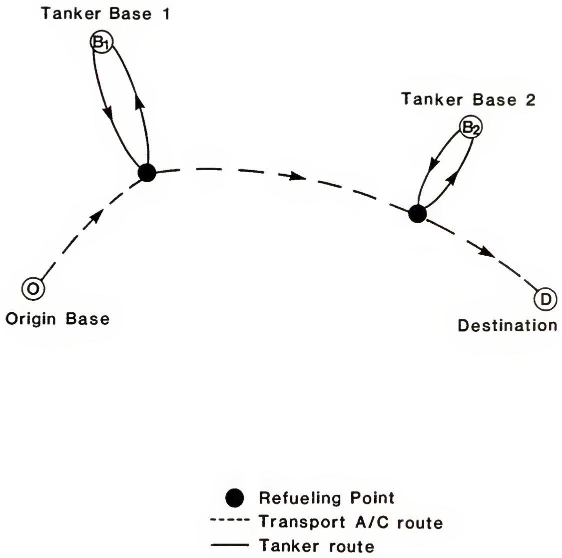


Figure 1-1: Illustration of aerial refueling and possible routes.

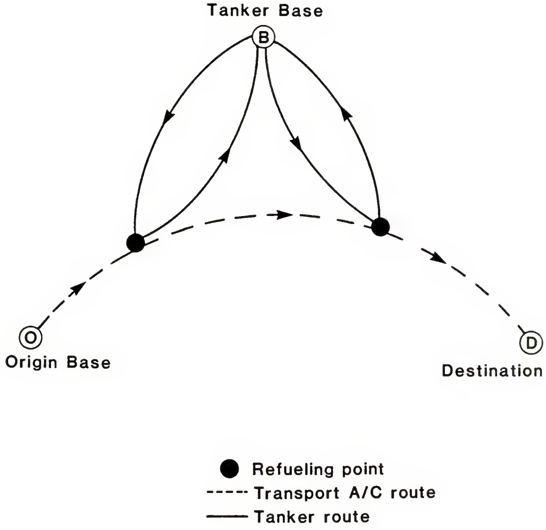


Figure 1-2: A second example of a possible route.

The general problem involves many variables, such as different types of A/C, the possibility of many tanker bases, and/or multiple refueling, etc. Even though a mathematical model for the general problem could be formulated, it would provide little insight. Rather, the approach taken has been to break the problem down into smaller problems, all of which are special cases of the general problem. By starting with the simplest problem and modelling them in increasing order of difficulty, one captures the essence of the general problem. The simpler problems also serve as building blocks for subsequent, more complicated ones. Figure 1-3 illustrates the various problems which are described below.

Problem (P1). There are N A/C, and any A/C used is required to fly directly to the destination. It is desired to determine how many A/C to use and how to divide the load among them.

Problem (P2). There is only one A/C with predetermined cargo weight, but it is unable to fly all the way to the destination without refueling. The location of the optimal refueling point and the initial fuel required for both the transport A/C and the tanker A/C need to be determined.

The first two problems are fundamental because no other problem can be solved without knowing first how to solve these two.

Problem (P3). This problem has the complexity of (P1) and (P2) combined. Here there are two A/C. The cargo weight that goes into each one needs to be determined and, depending on that weight, the A/C might need to refuel. Therefore, the location of the optimal refueling point of each A/C (if refueling is necessary) needs to be specified as well.

Problems and Solution Stages

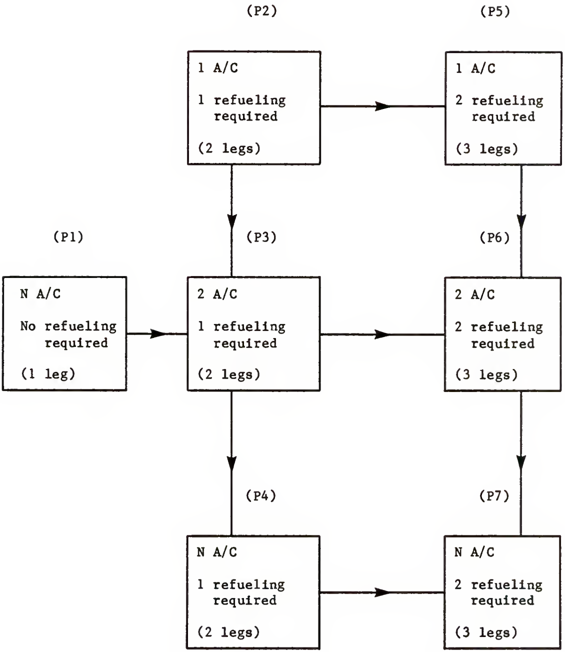


Figure 1-3: Different cases of the general problem and the solution stages.

Problem (P4). This problem is essentially the same as problem (P3) with N A/C instead of two.

Problem (P5). This problem is an extension of (P2) to include the possibility of refueling twice instead of once.

Problem (P6). This problem can be considered as an extension of (P5) to include two A/C instead of one or as an extension of (P3) with two refuelings instead of one.

Problem (P7). This problem is similar to (P6) but with N A/C. If this problem is solved, then the extension to the case of multiple refuelings should be easy and obvious.

In Figure 1-3, the arrows indicate the sequence that must be followed. In order to solve a certain problem, one must first solve all the problems leading to it.

Literature Survey

There exists no research in the reported literature that deals with this problem in its generality, i.e., (P7). However, by breaking it down into subproblems, one can identify a fundamental and important subproblem which involves one transport A/C where aerial refueling is required (i.e., P2). This subproblem concerns itself with finding the location of the refueling point such that the total fuel consumption is minimized [i.e., a location problem on a sphere (earth)]. The objective function is nonlinear in the distance, and the constraints (the A/C cannot be allowed to run out of fuel) depend on some of the decision variables.

A closely related problem is found in the literature of location theory, but it has a linear objective function. This problem is the

spherical Weber problem with maximum distance constraints. In the Weber problem, it is necessary to determine the optimal location of a source so as to minimize the sum of weighted distance (minisum objective) from the source to a finite number of given demand points whose location is known.

The spherical Weber problem on a unit sphere (without maximum distance constraints) can be stated mathematically as follows:

$$\text{Minimize } f(X) = \sum_{i=1}^{m} \beta_i \cdot d(X,P_i) \tag{1-1}$$

$$\text{S.T. } \| X \|^2 = 1, \text{ and } X \in E^3 \tag{1-2}$$

where

P_i = location of i^{th} demand point on the surface of the unit sphere, $i = 1, \ldots, m$

β_i = weight corresponding to the i^{th} demand point, $\beta_i \geq 0$, $i = 1, \ldots, m$

X = location of the source

$d(X,P_i)$ = distance between the source and the i^{th} demand point, measured along the surface of the sphere

$\|\cdot\|$ = the Euclidean norm.

Several researchers have investigated the nature of the optimal solution for the unconstrained spherical Weber problem (a summary of this can be found in Appendix C). Aly et al. [1] showed that the search for an optimal solution to this problem, where demand points are not located entirely on a great circle arc (see Appendix B for the definition of a great circle), can be restricted to the spherically convex hull of the demand points. Drezner [10] proved that if the demand points are located on a great circle, so is the optimal solution

point. Drezner and Wesolowsky [12] gave properties and results for the problem, some of which were shown to be invalid by Ayken [3]. He restated those properties and results and established additional ones. One of the restated results is that if all the demand points of a spherical Weber problem are included within a spherical disk of radius less than or equal to $\pi/4$, every minimum of the problem is a global minimum. An additional result was as follows: Suppose P_i, β_i, $i = 1, \ldots, m$ are the given points and their respective weights, where $m \geq 3$ and $\beta_i \geq 0$. If all the demand points are included within a spherical disk of radius $\pi/2$, and at least three demand points are not spherically colinear, the spherical Weber problem possesses a unique minimum. Furthermore, that minimum is in the spherical disk covering the demand points.

Katz and Cooper [20] gave a condition under which a demand point (θ_k, ϕ_k) is a local minimum. Let α_{ik} represent the shortest arc distance between P_i and P_k, where

$$\cos \alpha_{ik} = \cos \phi_i \cos \phi_k \cos(\theta_k - \theta_i) + \sin \phi_i \sin \phi_k. \qquad (1\text{-}3)$$

The demand point P_k is a local minimum if

$$\beta_k > (A_k^2 + B_k^2)^{1/2}, \text{ where} \qquad (1\text{-}4)$$

$$A_k = \sum_{\substack{i=1 \\ i \neq k}}^{m} \frac{\beta_i}{\sin \alpha_{ik}} [-\sin \phi_k \cos \phi_i \cos(\theta_k - \theta_i) + \cos \phi_k \sin \phi_i] \qquad (1\text{-}5)$$

$$B_k = \sum_{\substack{i=1 \\ i \neq k}}^{m} \frac{\beta_i}{\sin \alpha_{ik}} \cos \phi_i \sin(\theta_i - \theta_k) \qquad (1\text{-}6)$$

β_k = the weight associated with P_k.

This condition is similar to the so-called Kuhn modified gradient condition for the planar case (see Francis and White [16]).

Several solution procedures to the spherical Weber problem without maximum distance constraints have been developed. Litwhiler and Aly [22] solved the problem using two methods. The first, the Map Projection Algorithm (MPA), projects the current solution point to the plane using the azimuthal equidistance projection, solves the problem in the plane using euclidean distance, then transforms it back to the sphere. This is slow due to the back and forth transformation. The other method, Cyclic Search Algorithm (CSA), searches in a cycle in a direction orthogonal to a meridian along a great circle track and then along a meridian. However, the North Pole has to be transformed to the current solution point in order to find the great circle track. This algorithm is derivative free and is faster than the MPA.

Katz and Cooper [20] and Drezner and Wesolowsky [12] developed similar procedures for the Weber problem. Both are iterative procedures that use the derivative of the objective function. The difference between the two algorithms is in the functional form used to characterize the great circle distance.

Ayken [3] also developed two algorithms. The first is a derivative-free algorithm called Cyclic Meridian Parallel Search (CMPS). It is similar to the CSA of Litwhiler, but faster because no transformation of poles to the current point is needed. Also, CMPS can be used to minimize any type of function restricted to any surface on which the function and the point can be easily represented by curvilinear coordinates. The second algorithm is called the Geodesic Descent Algorithm (GDA). The search direction of GDA is the geodesic curve whose tangent

vector is the gradient direction of the objective function projected onto the tangent plane. GDA is analogous to the steepest descent algorithm used for the unconstrained problem in E^n, and GDA is faster than either CMS or MPA. Also, like CMS, GDA can be used to optimize any function defined on any surface for which the following conditions are met: it is possible to project a point lying outside the surface onto it; it is straightforward to find the path of the geodesic descent; and partial derivatives of the function are easily found. Note that the iterative scheme proposed by Katz and Cooper [20] could be considered as a fixed-step version of GDA as shown by Ayken [3].

All of the above algorithms solve the spherical Weber problem without maximum distance constraints. They converge to a local optima (which is also a global optima) if the search region is limited to a spherical disk of $\pi/4$ radius. Moreover, the optimal point may not be unique.

Now, we turn our attention to the constrained version of the problem. Constrained location problems in E^n have been studied by several researchers. Love [23] proposed a scheme to handle various types of spatial constraints which make use of convex programming and penalty function techniques. Love and Morris [24] studied the computational aspects of the solution of constrained multifacility location problems involving L_p distances using nonlinear optimization techniques. Hurter et al. [18] provided some properties of constrained location problems when the distance is derived from a norm.

Another version of the constrained location problem is one that includes the presence of a forbidden region in which no path is permitted to enter. This was studied by Katz and Cooper [6,19,21] using L_p distance.

Ayken [3] formulated the spherical Weber problem with maximum distance constraints in which the location of the sources is kept within a distance S_t of a fixed point U_t, where U_t may be one of the demand points. The maximum distance constraint on a unit sphere can be characterized as follows:

Let $I = \{X : ||X||^2 = 1, \text{Arc cos } (X'U_t) \leq S_t, X \varepsilon E^3\}$

Inequality arc cos $(X'U_t) \leq S_t$ is equivalent to the following two inequalities:

$$-X'U_t \leq - \cos (S_t) \tag{1-7}$$

$$X'U_t \leq 1 \tag{1-8}$$

The second inequality is redundant so the spherical Weber problem on a unit sphere with maximum distance constraints is formulated as follows:

$$\text{Min } f(X) = \sum_{i=1}^{m} \beta_i \cdot d(X,P_i) \tag{1-9}$$

$$\text{S.T. } ||X||^2 = 1 \tag{1-10}$$

$$-X'U_t \leq -D_t \qquad t = 1, \ldots, N \tag{1-11}$$

$$X \varepsilon E^3 \tag{1-12}$$

where $D_t = \cos (S_t)$.

Note that this formulation has a linear objective function and all points are expressed using cartesian coordinates rather than spherical coordinates. Ayken [3] solved this problem by first finding a vector tangent to the improving feasible direction. This is done by solving the following linear program subproblem.

Let $J = \{t : U_t' X = D_t\}$

$$\text{Minimize } f(x)' \cdot d \tag{1-13}$$

$$\text{S.T. } -U_t' d \leq -D_t \qquad t \in J \tag{1-14}$$

$$X'd = 0 \tag{1-15}$$

$$-1 \leq d_i \leq 1 \qquad i = 1,2,3 \tag{1-16}$$

where d_i = the i^{th} component of d. The component d_3 can be solved in terms of the other components of the vector d (i.e., d_1 and d_2) using equation (1-15). This reduces the subproblem and makes it possible to solve graphically. Before conducting a line search along the direction d, the maximum step size that can be taken from the current solution in this direction, without violating any constraint, has to be found. A formula is given for getting the maximum step size. Then a line search in this direction is conducted to find a new point. The process is repeated until convergence is reached.

Another version is the constrained maxmin or minmax facility location problem on the sphere. Several researchers have worked on this problem, including Drezner and Wesolowsky [13,14] and Drezner [11].

Darnell and Loflin [7,8] and Waite [28] attacked a simplified version of problem (P2) and (P5) in which there are unrestricted but finite number of fixed refueling points with different fuel costs. The objective is to fly the predetermined tour with minimum fuel cost. Darnell and Loflin [7,8] used linear programming to solve the deterministic version of this problem. While Waite [28] concentrated on the stochastic nature of this problem and solved it using stochastic dynamic programming.

Assumptions

In order to focus on the analytical elements of this study, the following simplifying assumptions are made.

1. The total cargo to be transported can be divided in any way desired, i.e., one can ignore the fact that the cargo is packaged in pallets and that one does not have the freedom to split the cargo load in any way except as the weight of the individual pallets permits. In essence, it is assumed that the total cargo weight is a continuous variable. The weight associated with the assignment of people to an aircraft will also be assumed to be a continuous variable.
2. Only the following limitation on the weight is to be considered: the gross weight of the A/C should not exceed the maximum take-off weight of the A/C (MTOW), [i.e., A/C empty weight + cargo weight + fuel weight $\leq$ MTOW].
3. There are no load balance or size restrictions. That is, any part of the cargo can fit into any part of the A/C as long as the maximum take-off weight restriction is not violated. This is reasonable in most cases because almost all pallets can fit anywhere in the transport A/C except possibly in the rear-most position of the A/C.
4. The earth is a perfect sphere with radius R, where

   R = 3,920 statute miles

   = 3,404 nautical miles.
5. The A/C follows the great circle route in flying from one point to another.
6. Weather at altitude (i.e., the jet stream) is assumed to be negligible. Inclusion of this factor would be relatively straightforward but would unnecessarily complicate the analytic development.

7. Aerial refueling takes a negligible amount of time. Thus, the region in which fuel transfer takes place is considered to be a single point.
8. Everything takes place at altitude. Thus, fuel consumed for take-off is ignored.
9. Only one type of transport A/C is available.

Scope of the Study

This study is concerned with analyzing the general A/C mid-air refueling problem described earlier in order to characterize and find its optimal solution. The limitations of this study are discussed in the previous section under "Assumptions."

The analysis approach used in this dissertation consists of breaking the general problem down into smaller and simpler problems that are addressed in order of increasing complexity up to the most general one. Starting with problems (P1) and (P2), the first five problems are solved in the sequence indicated by the arrows in Figure 1-3.

Chapter II contains derivation and characterization of the fuel consumption function and other related functions. These functions describe the relationship between fuel consumption, distance traveled, cargo weight, and initial fuel of a typical jet aircraft.

Chapter III contains a formulation of problem (P1). An exact solution for this problem, showing the optimal number of A/C to use, their optimal cargo weights and their initial fuel, is provided.

Chapter IV contains an extensive description and formulation of problem (P2). This problem requires the location of the optimal refueling point for one A/C to be determined together with the initial

fuel for both the transport and the tanker A/C. Spherical convexity of this problem is discussed and proved. A procedure to find the optimal solution is also provided.

Chapters V and VI present formulation and solution procedures for problems (P3) and (P4), respectively.

Chapter VII addresses (P5). A solution procedure based on the developments of Chapter IV is proposed and discussed. It is shown to converge at least locally and that it is stable. Computational results of some test problems support the conjecture that this solution procedure converges globally for real world problems. Finally, Chapter VIII summarizes the results obtained in this dissertation and provide directions for future research.

CHAPTER II

THE FUEL CONSUMPTION FUNCTION AND OTHER RELATED FUNCTIONS

Derivation of the Fuel Consumption Function and Other Related Functions

Here, mathematical representation of the various relationships between fuel consumption, distance traveled, cargo weight, and initial fuel for a typical jet A/C are derived.

The United States Air Force [29] provided raw data describing jet A/C performance at different gross weights, different altitudes, and different speeds. These raw data have to be transformed before they are useful. A sample of these data is shown in Figure 2-1 for a C-5A transport A/C flying at an altitude of 31,000 feet. Different curves are presented for different A/C weights, and they show the distance traveled per 1,000 pounds of fuel burned at a given speed and at that given gross weight (GW). Consider the points on the curve indicating 99 percent maximum specific range. It is assumed, without loss of generality, that the transport A/C will operate at those points since speed is maximized, while fuel consumption is only 1 percent greater than the minimum possible.

The distance traveled in miles per 1,000 pounds of fuel burned when the A/C gross weight is GW, denoted here as MPF (GW), is plotted against gross weight GW in Figure 2-2 using the 99 percent line in Figure 2-1. Least square linear and quadratic fits to the data were tried, and both fit well as seen in Figure 2-2.

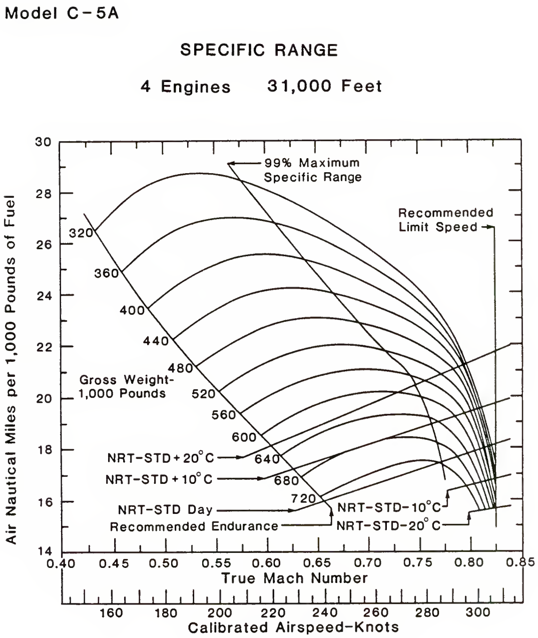


Figure 2-1: Data for C-5A aircraft.

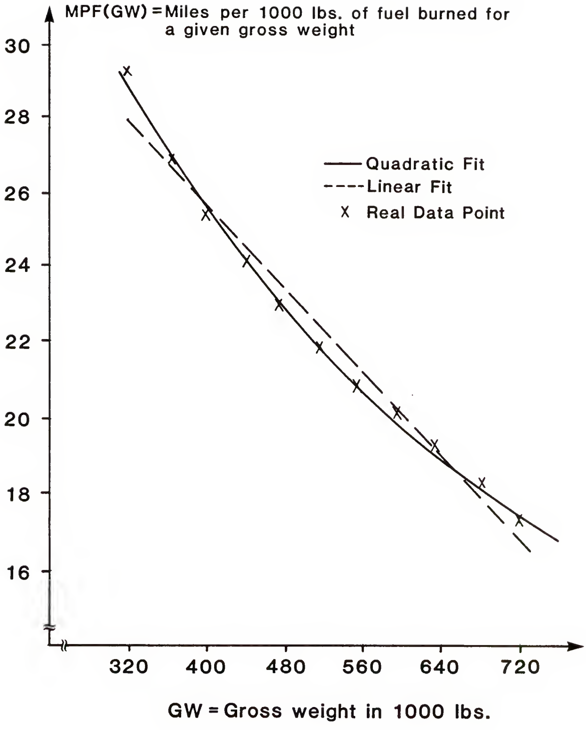


Figure 2-2: The MPF function versus GW.

For the linear fit,

$$MPF\ (GW) = a_o + a_1\ GW \tag{2-1}$$

where

$a_o = 36.2829$

$a_1 = -0.027$

and ρ = correlation coefficient = -0.9919.

For the quadratic fit,

$$MPF\ (GW) = b_0 + b_1\ GW + b_2 GW^2 \tag{2-2}$$

where

$b_o = 43.7616$

$b_1 = -0.0576$

$b_2 = 2.94 \times 10^{-5}$

and $\rho = -0.9983$.

The MPF function, in both forms, will be used to find the distance the A/C can travel when it has initial fuel g_o and cargo weight w_o. Also, it will be used to find out how much fuel is consumed when the A/C travels a given distance d, its cargo weight is w_o, and the initial fuel is g_o.

Note that at any moment during the flight,

$$GW = EW + w_o + f \tag{2-3}$$

where f = the present (instantaneous) amount of fuel and EW = A/C empty weight. During flight, f changes due to fuel burning; thus, GW changes while EW and w_o stay the same. So,

$$d\ GW = d\ f \tag{2-4}$$

The Range Function

Let $R(g_o,w_o)$ = the range of the tanker A/C when its initial fuel is g_o and cargo weight is w_o.

Initial Gross Weight = EW + w_o + g_o

Final Gross Weight = EW + w_o

then,

$$R(g_o,w_o) = \int_{EW + w_o}^{EW + w_o + g_o} MPF\ (GW)\ d\ GW$$

$$= \int_0^{g_o} MPF\ (GW)\ d\ f$$

$$= \int_0^{g_o} MPF\ (EW + w_o + f)\ d\ f \qquad (2\text{-}5)$$

(a) Using the linear fit for MPF, this results in

$$R(g_o,w_o) = (a_o' + a_1 w_o + \frac{a_1}{2} g_o)\ g_o \qquad (2\text{-}6)$$

$$\text{where } a_o' = a_o + a_1 EW \qquad (2\text{-}7)$$

(b) For a quadratic fit of MPF, this results in

$$R(g_o,w_o) = [b_o' + b_1' w_o + b_2 w_o^2 + (b_2 w_o + \frac{b_1}{2}) g_o + b_2 \frac{g_o^2}{2}]\ g_o \qquad (2\text{-}8)$$

$$\text{where } b_o' = b_o + b_1\ EW + b_2 EW^2 \qquad (2\text{-}9)$$

$$b_1' = b_1 + 2b_2 EW \qquad (2\text{-}10)$$

Figure 2-3 shows $R(g_o,w_o)$ for values of w_o and all values of g_o using both linear and quadratic fits.

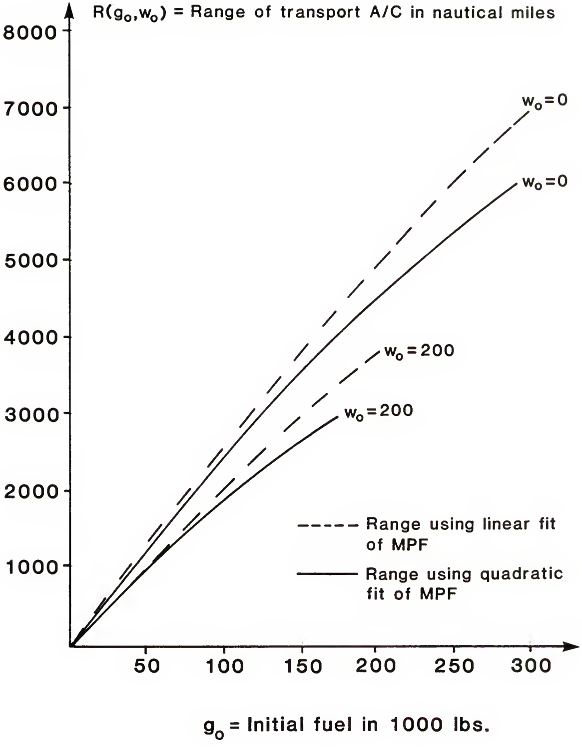


Figure 2-3: Range as a function of initial fuel and cargo weight.

The Fuel Consumption Function

Now, it is necessary to find the fuel consumed when the A/C flies a distance d [$d \leq R(g_o, w_o)$] with initial fuel g_o and cargo weight w_o.

Let

g_f = final amount of fuel left, and

$FC(g_o, w_o, d)$ = fuel consumed when initial fuel is g_o, cargo weight is w_o, and the distance flown is d,

then,

$$FC(g_o, w_o, d) = g_o - g_f \tag{2-11}$$

Thus, finding FC reduces to finding g_f, which can be accomplished by solving for g_f in the following equation:

$$d = \int_{g_f}^{g_o} MPF\,(EW + w_o + f)\,df \tag{2-12}$$

(a) For a linear fit of MPF, one has

$$\begin{aligned} d &= \int_{g_f}^{g_o} [a_o + a_1(EW + f + w_o)]\,df \\ &= a_o(g_o - g_f) + a_1(EW + w_o)(g_o - g_f) + \frac{a_1}{2}(g_o^2 - g_f^2) \end{aligned}$$

Solving for g_f results in

$$g_f = -\frac{a}{a_1} \pm \frac{\sqrt{(a + a_1 g_o)^2 - 2a_1 d}}{a_1} \tag{2-13}$$

Only the plus sign makes g_f physically possible, so

$$g_f = -\frac{a}{a_1} + \frac{\sqrt{(a + a_1 g_o)^2 - 2a_1 d}}{a_1} \tag{2-14}$$

Since the fuel consumed = $FC(g_o,w_o,d) = g_o - g_f$, then

$$FC(g_o,w_o,d) = g_o + \frac{a}{a_1} - \frac{\sqrt{(a + a_1 g_o)^2 - 2a_1 d}}{a_1} \tag{2-15}$$

where

$$a = a_o + a_1(EW + w_o) \tag{2-16}$$

Figure 2-4 shows a typical $FC(g_o,w_o,d)$ function for the linear case.

(b) The same procedure can be used to find $FC(g_o,w_o,d)$ for the case of a quadratic fit of MPF. The solution involves finding the root of a cubic equation by solving for g_o in the following equation:

$$\frac{b_2}{3} g_f^3 + \frac{C_2}{2} g_f^2 + C_1 g_f -$$

$$\left\{ C_1 g_o + \frac{C_2}{2} g_o^2 + \frac{b_2}{3} g_o^3 - d \right\} = 0 \tag{2-17}$$

and setting $FC(g_o,w_o,d) = g_o - g_f$

where

$$C_1 = b_o + b_1 (EW + w_o) + b_2 (EW + w_o)^2 \tag{2-18}$$

and

$$C_2 = 2\, b_2(EW + w_o) + b_1 \tag{2-19}$$

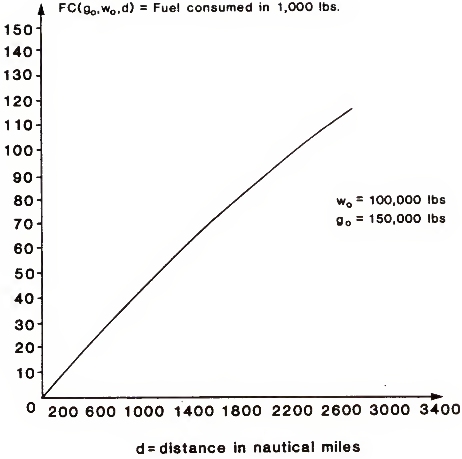


Figure 2-4: Fuel consumption as a function of distance for the linear case

Using the CRC Tables [27], the solution for g_f of equation (2-17) is found by letting

$$p = \frac{3}{2}\,\frac{C_2}{b_2}$$

$$q = 3\,\frac{C_1}{b_2}$$

$$r = \frac{-3}{b_2}\,\{C_1 g_o + \frac{C_2}{2}\,g_o^2 + \frac{b_2}{3}\,g_o^3 - d\}$$

$$e_1 = \frac{1}{3}\,(3\,q - p^2)$$

$$e_2 = \frac{1}{27}(2p^3 - 9pq + 27r)$$

$$e_3 = \sqrt{\frac{e_2^2}{4} + \frac{e_1^3}{27}}$$

$$A = \sqrt[3]{\frac{-e_2}{2} + e_3}$$

$$B = \sqrt[3]{\frac{-e_2}{2} - e_3}$$

Then,

$$g_f = A + B - \frac{p}{3}$$

The Fuel Requirement Function

Let $FN(w_o,d)$ = the exact amount of fuel needed to fly a distance d when the cargo weight is w_o. It is needed to find FN.

(a) For a linear fit of MPF, if the distance d to be flown and the cargo weight w_o are known in advance, then the amount of initial fuel g_o must be at least as big as the amount of fuel needed. If g_o is set to be equal to $FN(w_o,d)$, then we can use the range function $R(g_o,w_o)$ to solve for g_o. That is, set

$$R(g_o, w_o) = d = (a_o' + a_1 w_o + \frac{a_1}{2} g_o)\, g_o \qquad (2\text{-}6)$$

Solving for g_o results in the following equation:

$$FN(w_o, d) = g_o = -w_o - \frac{a_o'}{a_1} + \frac{\sqrt{(a_o' + a_1 w_o)^2 + 2a_1 d}}{a_1} \qquad (2\text{-}20)$$

(b) For the quadratic fit of MPF, the following equation is solved for g_o and then $FN(w_o,d)$ is set $= g_o$:

$$b_2 g_o^3 + (b_2 w_o + \frac{b_1}{2})\, g_o^2$$

$$+ (b_o' + b_1' w_o + b_2 w_o^2)\, g_o - d = 0. \qquad (2\text{-}21)$$

Again, using the CRC Tables [27], the solution for g_o of equation (2-21) is found by letting

$$p = w_o + \frac{b_1}{2b_2}$$

$$q = \frac{1}{b_2}(b_o' + b_1' w_o + b_2 w_o^2)$$

$$r = \frac{-d}{b_2}$$

$$e_1 = \frac{1}{3}(3q - p^2)$$

$$e_2 = \frac{1}{27}(2p^3 - 9pq + 27r)$$

$$e_3 = \sqrt{\frac{e_2^2}{4} + \frac{e_1^3}{27}}$$

$$A = \sqrt[3]{\frac{-e_2}{2} + e_3}$$

$$B = \sqrt[3]{\frac{-e_2}{2} - e_3}$$

Then,

$$g_o = A + B - \frac{p}{3}$$

Figures 2-5 and 2-6 show plots of FN (w_o,d) for the case of linear fit of MPF as a function of d and w_o, respectively. Table 2-1 summarizes the various fuel functions.

Characteristics of the Fuel Functions

The Fuel Requirement Function: FN (w,d)

$$FN\,(w,d) = -w - \frac{a'_o}{a_1} + \frac{\sqrt{(a'_o + a_1 w)^2 + 2a_1 d}}{a_1}$$

(1) The function FN (w,d) is a strictly convex and increasing function of the distance d for any given value of the cargo weight w.

Proof:

Take the first and second derivatives of FN with respect to d and note that w here is fixed at a constant value.

$$\frac{\partial\, FN}{\partial\, d} = \frac{1}{\sqrt{(a'_o + a_1 w)^2 + 2a_1 d}} \qquad > 0$$

$$\frac{\partial^2 FN}{\partial\, d^2} = \frac{-a_1}{[(a'_o + a_1 w)^2 + 2a_1 d]^{3/2}} \qquad > 0$$

because $a_1 < 0$. So, FN is strictly convex in d. Moreover, since the first derivative is always positive, then FN is an increasing function of d.

(2) The function FN is a strictly convex and increasing function of the cargo weight w for any given value of d. This can be shown in a straightforward fashion as in the previous case.

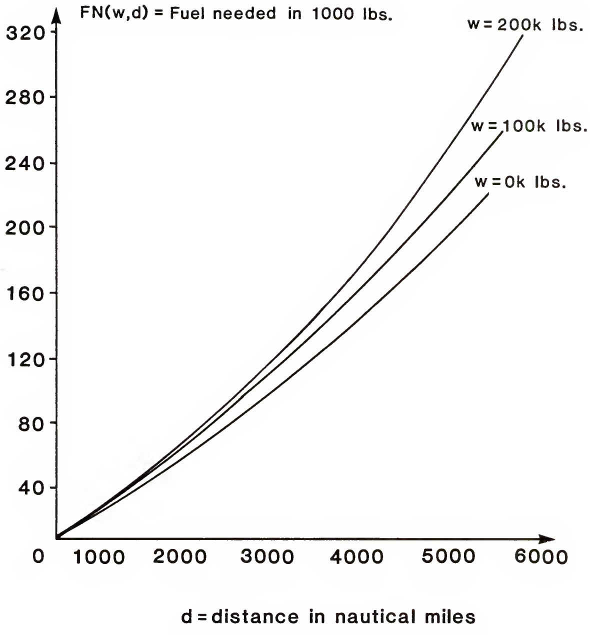


Figure 2-5: Fuel requirements as a function of distance and cargo weight.

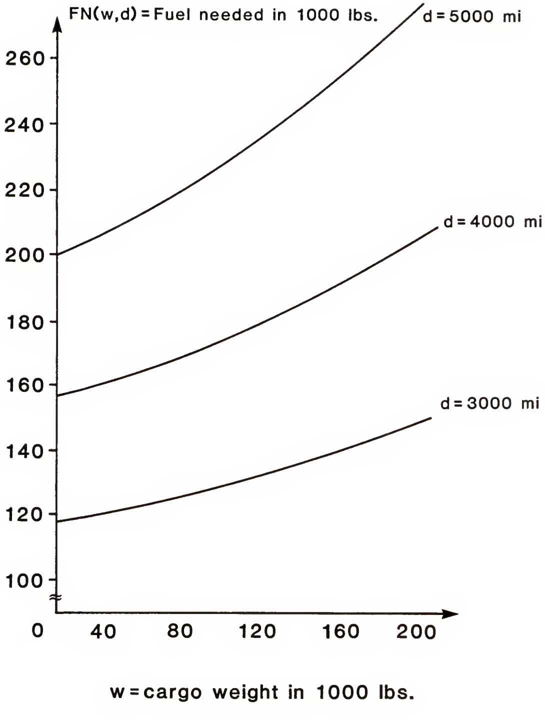


Figure 2-6: Fuel requirements as a function of cargo weight and distance.

TABLE 2-1: Summary of the fuel-related functions

Function Name	Function Value Using a Linear Fit for MPF	Function Value Using a Quadratic Fit for MPF
MPF(GW)	$a_o + a_1 \ GW$	$b_o + b_1 \ GW + b_2 \ GW^2$
$R(g_o,w_o)$ = Range	$(a_o' + a_1 w_o + \frac{a_1}{2} g_o)g_o$	$b_o' + b_1' \ w_o + b_2 w_o + (b_2 w_o + \frac{b_1}{2})g_o$ $+ b_2 \ g_o^2] g_o$
$FC(g_o,w_o,d)$ = Fuel Consumed	$g_o + \frac{a}{a_1} - \frac{\sqrt{(a + a_1 g_o)^2 - 2a_1 d}}{a_1}$	(Roots of a cubic equation) See text
$FN(w_o,d)$ = Fuel needed	$- w_o - \frac{a_o'}{a_1} + \frac{\sqrt{(a_o' + a_1 w_o)^2 + 2a_1 d}}{a_1}$	(Roots of a cubic equation) See Text

$a_o' = a_o + a_1 \ EW$ $\qquad$ $b_o' = b_o + b_1 \ EW + b_2 \ EW^2$

$a = a_o + a_1(EW + w_o) = a_o' + a_1 w_o$ $\qquad$ $b_1' = b_1 + 2b_2 \ EW$

We prove, in Appendix D (Theorem D-1), that if the distance d is measured along the surface of a sphere (earth), then the function FN (w,d) is spherically convex (s-convex) over a spherical disc of radius $\leq \pi/4$.

<u>The Fuel Consumption Function:</u> FC (g,w,d)

$$FC(g,w,d) = g + w + \frac{a_o'}{a_1} - \frac{\sqrt{[a_o' + a_1(g + w)]^2 - 2a_1 d}}{a_1}$$

(1) The function FC (g,w,d) is a concave and increasing function of the distance d for any given values of the initial fuel g and the cargo weight w.

(2) The function FC (g,w,d) is a convex and increasing function of w for any given values of g and d.

Again, both cases can be shown to be true in a fashion similar to that of FN(w,d).

CHAPTER III

PROBLEMS (P1)--MODEL AND SOLUTION

N A/C No Refueling required (1 leg)

Description and Formulation of (P1)

A total cargo weight of W is to be transported from one base to another where the distance between them is D_{OD}. There are N identical transport A/C available at the origin base "O." It is required that each of the $M \leq N$ A/C used fly directly to the destination "D."

The following variables need to be determined:

1. How many transport A/C to use, i.e., M = ?
2. How much cargo should be loaded into each A/C, i.e., w_n = ? for n = 1, 2, . . ., M.
3. How much fuel should be put in each A/C, i.e., g_n = ? for n = 1, 2, 3, . . ., M.

The objective is to minimize the total fuel consumption.

The following definitions are necessary:

EW = Transport A/C Empty Weight

MTOW = Maximum Take-off Weight of the transport A/C

F_{max} = Maximum fuel capacity of the transport A/C

$R_n(g_n, w_n)$ = Range of the n^{th} transport A/C when its initial fuel is g_n and cargo weight is w_n

$FN_n(w_n,d)$ = Exact amount of fuel needed by the n^{th} transport A/C to fly a distance d when its cargo weight is w_n.

Note that if $w_n = 0$ for some n, then that A/C will not be used. Hence, M can be determined easily once w_n's are known. Also, if $w_n > 0$ is known, then the amount of fuel needed ($FN(w_n,D_{OD})$) can be determined easily. Since all $M \leq N$ A/C used are flying directly, we could set the amount of initial fuel g_n to be equal to the amount of fuel needed, i.e.,

$$g_n = \begin{cases} FN_n\ (w_n,D_{OD}) & \text{if } w_n > 0 \\ 0 & \text{if } w_n = 0 \end{cases}$$

Thus, total fuel consumption would be equal to $\sum_{n=1}^{N} g_n$. Stating the problem mathematically,

$$\text{(P1) Minimize} \quad \sum_{n=1}^{N} g_n \qquad (3\text{-}1)$$

$$\text{S.T.} \quad \sum_{n=1}^{N} w_n = W \qquad (3\text{-}2)$$

$$g_n + w_n \leq (MTOW - EW) \qquad \forall\ n \qquad (3\text{-}3)$$

$$0 \leq g_n \leq F_{max} \qquad \forall\ n \qquad (3\text{-}4)$$

$$g_n = \begin{cases} FN_n\ (w_n,D_{OD}) & \text{if } w_n > 0 \\ 0 & \text{if } w_n = 0 \end{cases} \qquad \forall\ n \qquad (3\text{-}5)$$

$$w_n \geq 0 \qquad \forall\ n \qquad (3\text{-}6)$$

This problem is nonlinear because equation (3-5) is nonlinear.

Solution of (P1)

The following two theorems tell us the optimal number of A/C to use and their optimal cargo weights.

Theorem 3-1

Given that M A/C are to be used, the total cargo load should be divided equally among those M A/C, i.e.,

$$w_n^* = \begin{cases} \frac{W}{M} & n = 1, 2, \ldots, M \\ 0 & n = M + 1, \ldots, N \end{cases}$$

Proof of Theorem 3-1. Since M A/C will be used, then

$$w_n > 0 \qquad \text{for } n = 1, 2, \ldots, M$$

and $w_n = 0 \qquad$ for $n = M + 1, \ldots, N$

Then from equation (3-5), it must be true that

$$g_n = FN_n(w_n, D_{OD}) \qquad \text{for } n = 1, 2, \ldots, M$$

Thus, the problem in this case is stated as

$$\text{(T1)} \quad \text{Min} \sum_{n=1}^{M} g_n = \sum_{n=1}^{M} FN_n\,(w_n, D_{OD})$$

$$\text{S.T.} \quad \sum_{n=1}^{M} w_n = W$$

$$w_n > 0 \qquad n = 1, \ldots, M$$

$$g_n + w_n \leq (MTOW - EW) \qquad n = 1, \ldots, M \qquad (3\text{-}3)$$

$$0 < g_n \leq F_{max} \qquad n = 1, \ldots, M \qquad (3\text{-}4)$$

The last two equations will be ignored for the moment. If the optimal solution to the resulting relaxed problem satisfies them, the solution is optimal to the original problem (T1). The relaxed problem (RT1) is as follows:

$$\text{(RT1)} \quad \text{Min} \quad \sum_{n=1}^{M} FN_n\,(w_n, D_{OD}) \tag{3-9}$$

$$\text{S.T.} \quad \sum_{n=1}^{M} w_n = W \tag{3-10}$$

$$w_n > 0 \qquad n = 1, 2, \ldots, M \tag{3-11}$$

Note that the objective function is the sum of strictly convex functions (as shown before). In addition, the feasible region is a convex set. To solve this problem, we can either eliminate equation (3-10) by solving for any of the w_n's, or use the Lagrange multiplier method.

The optimal solution for (RT1), using either method, is

$$w_n^* = \frac{W}{M} \qquad \text{for } n = 1, 2, \ldots, M \tag{3-12}$$

$$\text{and } g_n^* = FN\,(\tfrac{W}{M}, D_{OD}) \qquad \text{for } n = 1, 2, \ldots, M \tag{3-13}$$

What does this imply regarding problem (T1)? If this solution of (RT1) is feasible to (T1) [i.e., it satisfies eqs. (3-3) and (3-4)], then it is the unique optimal solution for (T1) because the objective function is strictly convex. Otherwise there is no feasible solution to (T1). To prove the last statement, look at eqs. (3-3) and (3-4). By assumption, at least one of them is not satisfied, i.e., we will have

$$FN\,(\tfrac{W}{M}, D_{OD}) + \tfrac{W}{M} > MTOW - EW \qquad \text{or}$$

$$FN\,(\tfrac{W}{M}, D_{OD}) > F_{max}$$

Since it is not feasible to have all w_n's equal to $\frac{W}{M}$, then any other feasible solution must have

$$w_i > \frac{W}{M} \text{ for some } i$$

But having $w_i > \frac{W}{M}$ means having a larger g_i than before, i.e.,

$$w_i > \frac{W}{M} \rightarrow g_i = FN\ (w_i, D_{OD}) > FN\ (\frac{W}{M}, D_{OD})$$

Thus, we would have

$$FN\ (w_i, D_{OD}) + w_i > FN\ (\frac{W}{M}, D_{OD}) + \frac{W}{M} > MTOW - EW$$

or

$$FN\ (w_i, D_{OD}) > FN\ (\frac{W}{M}, D_{OD}) > F_{max}$$

which means no other feasible solution to (T1) exists.

In conclusion, if it is feasible to use M A/C, then

$$w_n^* = \frac{W}{M} \quad \text{for } n = 1, 2, \ldots, M \text{ and } g_n^* = FN\ (\frac{W}{M}, D_{OD})$$

Theorem 3-2

Let M_{min} = the minimum feasible number of A/C that can be used.

If $\frac{a_o'}{-2a_1} - \frac{D_{OD}}{a_o'} \geq \frac{W}{M_{min}}$, then the minimum feasible number of A/C results in the lowest total fuel cost.

Proof of theorem 3-2. Here we need to show that the cost is an increasing function of the number of A/C used. First, note that if M A/C are used, then

$$w_n^* = \begin{cases} \frac{W}{M} & n = 1, 1, \ldots, M \\ 0 & n = M + 1, \ldots, N \end{cases}$$

from theorem 3-1.

Let TF(M) = the total fuel cost when M A/C are used ($M \geq 1$ and an integer). Thus, TF(M) is a discrete function.

$$TF(M) = \sum_{n=1}^{M} g_n = \sum_{n=1}^{M} FN_n \quad (w_n, D_{OD}) = M * FN\left(\frac{W}{M}, D_{OD}\right)$$

$$= -W - M\frac{a_o'}{a_1} + \frac{M}{a_1}\sqrt{(a_1\frac{W}{M} + a_o')^2 + 2a_1 D_{OD}} \qquad (3\text{-}13)$$

Let TF1(M) = TF(M) where $M \geq 1$ and continuous. Here, TF1(M) is the continuous version of TF(M). That is, it includes the value of the function for integer as well as noninteger values of M. Also both functions should have the same shape, (i.e., if TF1(M) is increasing, so is TF(M)). Thus, we need only to show that TF1(M) is an increasing function of M, or equivalently that the first derivative of TF1(M) is nondecreasing.

$$\frac{dTF1(M)}{dM} = \frac{-a_o'}{a_1}\left[1 - \frac{(a_1\frac{W}{M} + a_o') + \frac{2a_1}{a_o'} D_{OD}}{\sqrt{(a_1\frac{W}{M} + a_o')^2 + 2a_1 D_{OD}}}\right] \qquad (3\text{-}14)$$

Note that $a_o' > 0$ and $a_1 < 0$.

Now $\frac{dTF1(M)}{dM} \geq 0$ if $\frac{a_o'^2}{-a_1} \geq 2(a_o'\frac{W}{M} + D_{OD})$

$$\text{or equivalently if } \frac{a_o'}{-2a_1} - \frac{D_{OD}}{a_o'} \geq \frac{W}{M} = w_n \qquad (3\text{-}15)$$

Certainly if this is true for $M = M_{min}$, it would be true for $M \geq M_{min}$.

Q.E.D.

Let's see the implication of this for the C-5A and a real-world problem. The left hand side of inequality (3-15) is minimum when D_{OD} is maximum and still s-convex, i.e., max D_{OD} = 6,220 nautical miles (or $\pi/4$ in Radian).

For the C-5A we have

$a_1 = -0.027$, $a_o = 36.283$, EW = 320 k lbs

so $a_o' = a_o + a_1$ EW = 27.643

Thus, we need to have

$$286 \text{ k lbs} \geq \frac{W}{M} = w_n$$

But from the physical restriction on the C-5A we know that $w_n \leq 200$ k lbs. Thus the above condition is always satisfied for the C-5A. Moreover M_{min} is the smallest positive integer M that satisfies the following two conditions:

$$FN\left(\frac{W}{M}, D_{OD}\right) \leq \text{Min}\left\{F_{max}, MTOW - EW - \frac{W}{M}\right\}$$

and

$$\frac{W}{M} \leq \quad w_{max} \quad \text{where}$$

w_{max} = maximum allowable cargo weight due to structural weight limitation

Figure 3-1 shows the total fuel cost, TF(M), as a function of the number of A/C used.

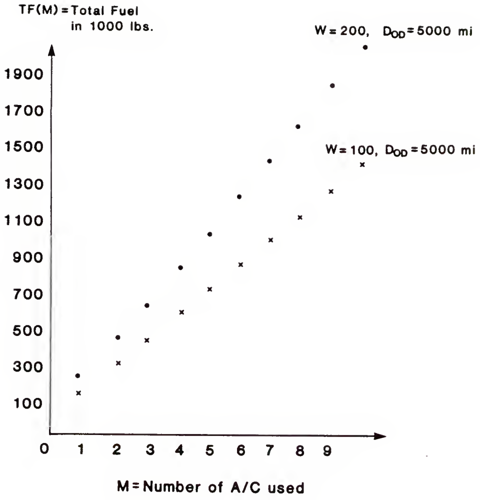


Figure 3-1: Total fuel cost versus number of A/C used.

CHAPTER IV

PROBLEM (P2)--MODEL AND SOLUTION

1 A/C 1 Refueling required (2 legs)

Problem Description

Consider the situation where there is one transport A/C with cargo weight w_o. The fuel required to travel a distance D_{OD}--the distance between the origin and destination--is FN (w_o, D_{OD}). But, the maximum fuel the transport A/C can carry is $g_{max}(w_o) = \text{Min}\{F_{max}, \text{MTOW} - \text{EW} - w_o\}$. So, if FN $(w_o, D_{OD}) > g_{max}(w_o)$ or equivalently if $R[g_{max}(w_o), w_o] < D_{OD}$, then the transport A/C must refuel en route in order to reach its destination. Now, assume there is a third base that has a tanker A/C ready to refuel the transport A/C in mid-air. The maximum fuel the tanker can carry is H_{max}. In this case both A/C would fly to a point (θ, ϕ) where they meet. There, some fuel would be transferred from the tanker to the transport A/C. The tanker then would return to its base, and the transport A/C would continue to its destination. The objective is to determine the location of the refueling point so that the total fuel requirement for all A/C is minimized. There are some limitations on the location of the refueling point. To see this, let

$d_1(\theta, \phi)$ = the distance between the origin base (θ_0, ϕ_0) and the refueling point (θ, ϕ)

$d_2(\theta,\phi)$ = the distance between the refueling point (θ,ϕ) and the destination (θ_D,ϕ_D)

$d_3(\theta,\phi)$ = the distance between the tanker base (θ_B,ϕ_B) and the refueling point (θ,ϕ)

$R\ (g_o,w_o)$ = the range of the transport A/C when its initial fuel is g_o and cargo weight is w_o

$g_{max}(w_o)$ = the maximum fuel the transport A/C can carry when its cargo weight is w_o and still be able to take-off

= $\min\{F_{max}, \text{MTOW} - \text{EW} - w_o\}$

GW_{max} = the maximum allowed gross weight of a transport A/C

$G_{max}(w_o)$ = the maximum fuel the transport can carry when it is in the air, given that its cargo weight is w_o

= $\min\{F_{max}, GW_{max} - \text{EW} - w_o\}$

$R^*(h_o,w)$ = the range of the tanker A/C when its initial fuel is h_o and the cargo weight is w (here, $w = 0$)

$FN^*(w,d)$ = exact amount of fuel needed by the tanker to fly a distance d when its cargo weight is w (usually $w = 0$)

$FC^*(h_o,w,d)$ = the fuel consumed by the tanker in flying a distance d when its initial fuel is h_o and cargo weight is w (usually $w = 0$)

f_1 = $f_1(\theta,\phi)$

= the fuel consumed by the transport A/C in flying from (θ_0,ϕ_0) to (θ,ϕ)

= $FC(g_o,w_o,d_1)$

f_2 $= f_2(\theta, \phi)$

= the fuel needed by the transport A/C to go from (θ, ϕ) to (θ_D, ϕ_D)

$= FN(w_o, d_2)$

f_3 $= f_3(\theta, \phi)$

= the fuel consumed by the tanker in going from (θ_B, ϕ_B) to (θ, ϕ)

$= FC^* (h_o, 0, d_3)$

f_4 $= f_4(\theta, \phi)$

= the fuel needed by the tanker to go from (θ, ϕ) back to (θ_B, ϕ_B)

$= FN^* (0, d_3)$

There are several conditions that must be satisfied by the location of the refueling point. First, the distance between the origin base and the refueling point should not exceed the maximum range of the transport A/C, i.e.,

$$d_1(\theta, \phi) \leq R[g_{max}(w_o), w_o] \tag{4-1}$$

Second, refueling has to be done such that the transport A/C is able to reach its destination from the refueling point, i.e.,

$$d_2(\theta, \phi) \leq R[G_{max}(w_o), w_o] \tag{4-2}$$

Third, the tanker should be able to make a round trip safely, i.e.,

$$d_3(\theta, \phi) \leq \frac{1}{2} R^*(H_{max}, 0) \tag{4-3}$$

Also, there are other conditions that must be satisfied by g_o and h_o, the initial fuel for the transport and the tanker, respectively. In

fact, whether or not g_o and h_o are feasible depends upon the location of the refueling point. This is because g_o must be at least equal to the amount of fuel needed to fly from the origin to the refueling point, i.e.,

$$FN(w_o, d_1) \leq g_o\ (\theta, \phi) \leq g_{max}\ (w_o) \tag{4-4}$$

Similarly, h_o should permit the tanker to make a trip from its base to (θ, ϕ) and back, i.e.,

$$FN^*\ (0, 2d_3) \leq h_o\ (\theta, \phi) \leq H_{max} \tag{4-5}$$

Finally, at the refueling point, the amount of fuel left in the tanker is $(h_o - f_3)$, and in the transport is $(g_o - f_1)$. To fly from the refueling point to the destination, the transport needs an amount of fuel equal to f_2. Thus, the amount of fuel to be transferred from the tanker to the transport A/C is $= f_2 - (g_o - f_1) = f_1 + f_2 - g_o$. The amount of fuel left in the tanker after transferring $= (h_o - f_3)$ - amount transferred $= (h_o - f_3) - (f_1 + f_2 - g_o) = h_o + g_o - f_1 - f_2 - f_3$. This amount has to be greater than or equal to the amount of fuel needed by the tanker to fly back to its base, or:

$$h_o + g_o - f_1 - f_2 - f_3 \geq f_4, \text{ or}$$

$$h_o + g_o \geq \sum_{i=1}^{4} f_i \tag{4-6}$$

<u>Formulation of (P2)</u>

The problem can be stated mathematically as follows: find (θ, ϕ), g_o and h_o that will

$$\text{(P2) Minimize } V(\theta, \phi) = \sum_{i=1}^{4} f_i(\theta, \phi)$$

$$\text{S.T. } d_1(\theta,\phi) \leq R[g_{max}(w_o),w_o] \tag{4-1}$$

$$d_2\ (\theta,\phi) \leq R[G_{max}(w_o),w_o] \tag{4-2}$$

$$d_3(\theta,\phi) \leq \frac{1}{2}\ R^*(H_{max},0) \tag{4-3}$$

$$FN(w_o,d_1) \leq g_o \leq g_{max}(w_o) \tag{4-4}$$

$$FN^*(0,2d_3) \leq h_o \leq H_{max} \tag{4-5}$$

$$(h_o + g_o) \geq \sum_{i=1}^{4} f_i(\theta,\phi) \tag{4-6}$$

$$0 \leq \theta \leq 2\pi \tag{4-7}$$

$$0 \leq \phi \leq \pi \tag{4-8}$$

A few remarks are in order. First, the intersection of the first three inequality constraints ((4-1), (4-2) and (4-3)) represent a region inside which all feasible refueling points must lie. Figure 4-1 shows an example of such a region. Note that even though all the feasible points are included in this region, not all the points in this region are necessarily feasible. For any point in this region to be feasible, there must exist values for its associated g_o and h_o that satisfy inequality (4-6) which can be rewritten as

$$[g_o(\theta,\phi) - f_1(\theta,\phi)] + [h_o(\theta,\phi) - f_3(\theta,\phi)] \geq f_2(\theta,\phi) + f_4(\theta,\phi) \tag{4-9}$$

The LHS of (4-9) is largest when both g_o and h_o attain their maximum values. From this, it can be concluded that any feasible point must satisfy (4-1), (4-2), (4-3) and the following constraint:

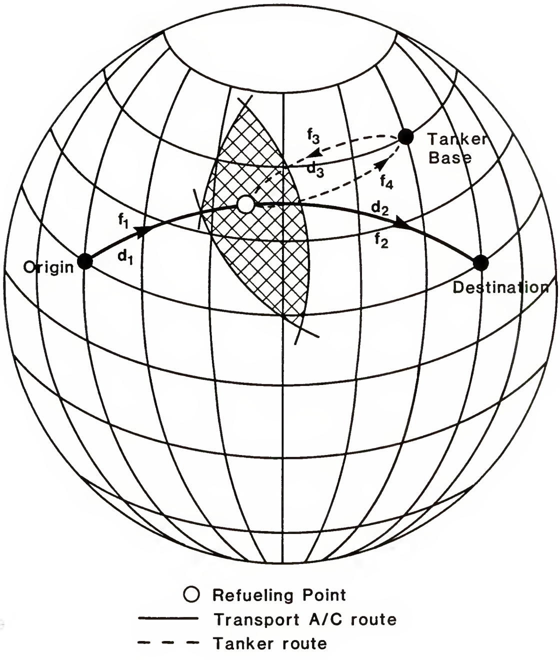


Figure 4-1: The region inside which all feasible refueling points lie.

$$\{g_{max}(w_o) - FC[g_{max}(w_o),w_o,d_1)\} + [H_{max} - FC^*(H_{max},0,d_3)]$$

$$\geq f_2(\theta,\phi) + f_4(\theta,\phi) \tag{4-10}$$

which is derived from (4-4), (4-5), and (4-6). Therefore, it is easy to check for the feasibility of any point inside this region. Also, note that the RHS of (4-6) is, in fact, the objective function. Therefore, if we start with a feasible point and find another one, inside the region described above, that has less cost, the second point is automatically a feasible point.

The second remark also is concerned with constraint (4-6). Since values of g_o and h_o greater than needed will increase the fuel consumption unnecessarily, we can restrict our attention to the case where

$$g_o(\theta,\phi) + h_o(\theta,\phi) = \sum_{i=1}^{4} f_i(\theta,\phi) \tag{4-11}$$

This means that there is only one value of $h_o(\theta,\phi)$ for every choice of $g_o(\theta,\phi)$. This value can be calculated from (4-11) as

$$h_o(\theta,\phi) = FN^*[h_f(g_o),d_3] + h_f(g_o) \tag{4-12}$$

where

$$h_f(g_o) = f_2(\theta,\phi) + f_4(\theta,\phi) - [g_o(\theta,\phi) - f_1(\theta,\phi)] \tag{4-13}$$

This means that we have to worry only about selecting for each point the best value of $g_o(\theta,\phi)$ (denoted by $g_o^*(\theta,\phi)$) that will result in the lowest total fuel cost for that point. Consequently, $g_o^*(\theta,\phi)$ is the solution to the following NLP subproblem in one dimension (g_o):

$$\text{(SP1)}\quad \text{Minimize } g_o + FN^*[h_f(g_o),d_3] + h_f(g_o) \tag{4-14}$$

$$\text{S.T.} \quad FN\ (w_o, d_1) \leq g_o \leq g_{max}(w_o) \tag{4-4}$$

$$FN^*[h_f(g_o), d_3] + h_f(g_o) \leq H_{max} \tag{4-15}$$

where $h_f(g_o)$ is defined by (4-13). This problem is convex in g_o and can be easily solved using Golden Section Search or Fibonacci Search if we combine (4-2) and (4-15) into the following single constraint:

$$g_{min}(\theta, \phi) \leq g_o(\theta, \phi) \leq g_{max}(w_o), \tag{4-16}$$

where

$$g_{min}(\theta, \phi) = \begin{cases} FN\ (w_o, d_1) & \text{if } g_f(\theta, \phi) \leq 0, \\ FN\ (g_f + w_o, d_1) + g_f & \text{if } g_f(\theta, \phi) \geq 0, \end{cases} \tag{4-17}$$

and

$$g_f(\theta, \phi) = f_2(\theta, \phi) + f_4(\theta, \phi) - [H_{max} - FC^*(H_{max}, 0, d_3)] \tag{4-18}$$

The third remark is that the great circle distance does not satisfy the usual convexity definition in E^n. However, this definition has been modified and extended to accommodate the situation where distance and set are defined on the surface of a sphere [1,3,10,12]. The following definitions, which describe spherical convexity, are needed to follow the discussion relative to the convexity of the objective function and the feasible region. (The interested reader is referred to Appendix C for a complete list of properties and results of spherical convexity.)

Definition 1

A set of points on a sphere is said to be spherically convex (s-convex set) if, for any two points of the set, the whole shorter great circle arc connecting them is included in the set [12].

Note that on a sphere the largest convex set, other than a whole sphere, is a hemisphere.

Definition 2

A spherically convex combination of two points r_1 and r_2 is defined as the point $r = \rho(r_1, r_2, \lambda)$ that lies on the shorter great circle arc between r_1 and r_2, such that the great circle distance between r_1 and r is $\lambda * d(r_1, r_2)$ for $0 \leq \lambda \leq 1$ [12].

Definition 3

A function $f(r)$ over an s-convex set is said to be a spherically convex function (s-convex function) if for every $0 \leq \lambda \leq 1$:

$$f(\rho(r_1, r_2, \lambda)) \leq \lambda f(r_1) + (1 - \lambda) f(r_2)$$

for any element r_1 and r_2 of the set [12].

Another important result due to Drezner and Weslowsky [12] says that the distance from a given point r is a s-convex function within a circle of radius $\pi/2$ and center r. Since the right hand side of constraints (4-1), (4-2) and (4-3) are usually much less that $\pi/2$, each of these constraints represent a s-convex set. The region resulting from their intersection will also be an s-convex set. Therefore Problem (P2), without constraint (4-10), will have a s-convex feasible region.

Our fourth remark is related to Aly et al. [1] who showed that the search for an optimal solution to the spherical Weber problem can be restricted to the spherical convex hull of the demand points. We have extended this result to the case at hand (see Appendix D) to show that

the search for the optimal refueling point can be restricted to the spherically convex hull of the three base points (the origin, the destination, and the tanker base).

Taking the first and fourth remarks together, we can see that the search for the optimal refueling point can be restricted to the intersection of the region described by (4-1), (4-2), and (4-3) with the spherically convex hull of the three bases.

Finally, the fifth remark concerns the objective function. It is a nonlinear function of the spherical distance (the great circle distance). Using the fuel function derived earlier, it is seen that

$$f_1 = g_o + \frac{a}{a_1} - \frac{\sqrt{(a + a_1 g_o)^2 - 2a_1 d_1}}{a_1}$$

$$f_2 = -\frac{a}{a_1} + \frac{\sqrt{a^2 + 2a_1 d_2}}{a_1}$$

$$f_3 = h + \frac{a'_o}{a_1} - \frac{\sqrt{(a'_o + a_1 h_o)^2 - 2a_1 d_3}}{a_1}$$

$$f_4 = \frac{-a'_o}{a_1} + \frac{\sqrt{[a'_o]^2 + 2a_1 d_3}}{a_1}$$

$$\sum_{i=1}^{4} f_i = h_o + g_o - \frac{1}{a_1}\left\{\begin{array}{l}\sqrt{(a+a_1 g_o)^2 - 2a_1 d_1} - \sqrt{a^2+2a_1 d_2} \\ + \sqrt{(a'_o+a_1 h_o)^2 - 2a_1 d_3} - \sqrt{(a'_o)^2 + 2a_1 d_3}\end{array}\right\}$$

The way the objective function is written, it is difficult to verify its s-convexity either by using the second derivative test or by

applying the definition of a s-convex function directly. Nevertheless, s-convexity of the objective function can be shown by rewriting the objective function using some s-convex function like $FN(\cdot,\cdot)$. To do this, define for any point (θ,ϕ) the following:

$WFL[g_o(\theta,\phi),d_1]$ = weight of the fuel left in the transport A/C when it flies a distance d_1 (from the origin to (θ,ϕ)) given that its initial fuel is $g_o(\theta,\phi)$

Thus, $WFL\,[g_o(\theta,\phi),d_1] = g_o(\theta,\phi) - FC[g_o(\theta,\phi),w_o,d_1]$ (4-19)

where $FN(w_o,d_1) \leq g_o(\theta,\phi) \leq g_{max}(w_o)$. (4-4)

$WFL(\cdot,\cdot)$ is s-convex because $FC(\cdot,\cdot,\cdot)$ is s-concave.

Let $WFL_{max}(d_1) = \underset{g_o}{\text{Max}}\ WFL\,[g_o(\theta,\phi),d_1]$

$$= WFL\,[g_{max}(w_o),d_1]. \qquad (4\text{-}20)$$

Note that

$$0 \leq WFL\,[g_o(\theta,\phi),d_1] \leq WFL_{max}(d_1), \qquad (4\text{-}21)$$

and that $WFL_{max}(.)$ is s-convex also.

Since $g_o^*(\theta,\phi)$ is the best value of $g_o(\theta,\phi)$, then using the s-convex functions FN and WFL, the elements of the objective function can be rewritten as

$$f_1(\theta,\phi) = FN\,\{w_o + WFL[g_o^*(\theta,\phi),d_1],d_1\} \qquad (4\text{-}22)$$

$$f_2(\theta,\phi) = FN(w_o,d_2) \qquad (4\text{-}23)$$

$$f_4(\theta,\phi) = FN^*\,(0,d_3) \qquad (4\text{-}24)$$

$$f_3(\theta,\phi) = FN^*\,\{f_2 + f_4 - WFL[g_o^*(\theta,\phi),d_1],d_3\} \qquad (4\text{-}25)$$

and the total fuel consumption $= V\,(\theta,\phi) = \sum_{i=1}^{4} f_i(\theta,\phi)$.

Both FN $(\cdot,\cdot)$ and FN^* $(\cdot,\cdot)$ are convex and increasing functions of their arguments, which, in turn, are s-convex functions. Therefore, the resulting objective function is s-convex [17,26]. In Appendix D, we present another method to prove s-convexity of the objective function. There, we give a sufficient condition for the s-convexity which depends on the distance between the points of the feasible region and the destination. This condition is arrived at using the worst case method.

Motivation for the Solution Procedure of (P2)

For this problem, the optimal point (θ^*,ϕ^*) and its optimal $g_o^*(\theta^*,\phi^*)$ (and consequently, its optimal $h_o^*(\theta^*,\phi^*)$) are to be found. Since $g_o^*(\theta,\phi)$ depends upon (θ,ϕ) and we do not know this in advance (remember, we have to solve (SP1) to find that out!), it is difficult to use (4-22), (4-23), (4-24) and (4-25) directly. That is, we cannot search for both (θ^*,Φ^*) and $g_o^*(\theta^*,\phi^*)$ simultaneously. However, we can overcome this by creating an upper bound function that can be improved from one iteration to another until its minimum value coincides with the minimum of the original function. To see this let

$$\alpha\,(\theta,\phi) = \frac{WFL\ [g_o(\theta,\phi),d_1]}{WFL_{max}(d_1)} \tag{4-26}$$

then,

$$0 \leq \alpha\,(\theta,\phi) \leq 1 \tag{4-27}$$

for all (θ,ϕ).

Thus, $g_o^*(\theta,\phi)$ would result in $\alpha^*(\theta,\phi)$, where $\alpha^*(\theta,\phi)$ is the best value of $\alpha\,(\theta,\phi)$, i.e.,

$$\alpha^*(\theta,\phi) = \frac{WFL[g_o^*(\theta,\phi),d_1}{WFL_{max}(d_1)} \tag{4-28}$$

Also, from (3-38), we have

$$WFL\ [g_o(\theta,\phi),d_1] = \alpha(\theta,\phi) \cdot WFL_{max}(d_1) \tag{4-29}$$

The element of the objective function can be rewritten using $[\alpha^*(\theta,\phi) \cdot WFL_{max}(d_1]$ instead of $WFL[g_o^*(\theta,\phi),d_1]$ as:

$$f_1(\theta,\phi) = FN\ [w_o + \alpha^*(\theta,\phi) \cdot WFL_{max}(d_1),d_1]$$

$$f_2(\theta,\phi) = FN\ (w_o,d_2)$$

$$f_4(\theta,\phi) = FN^*\ (0,d_3)$$

$$f_3(\theta,\phi) = FN^*\ [f_2 + f_4 - \alpha^*(\theta,\phi) \cdot WFL_{max}(d_1),d_3]$$

Now setting $\alpha(\theta,\phi) = \alpha$ for all points, and denoting the resulting function by $UB(\theta,\phi,\alpha)$, one gets

$$f_1'(\theta,\phi) = FN\ \{w_o + \alpha \cdot WFL_{max}(d_1),d_1\} \tag{4-30}$$

$$f_2'(\theta,\phi) = f_2(\theta,\phi) = FN\ (w_o,d_2) \tag{4-31}$$

$$f_4'(\theta,\phi) = f_4(\theta,\phi) = FN^*\ (0,d_3) \tag{4-32}$$

$$f_3'(\theta,\phi) = FN^*\ \{f_2 + f_4 - \alpha \cdot WFL_{max}(d_1),d_3\} \tag{4-33}$$

where $0 \leq \alpha \leq 1$

and $$UB(\theta,\phi,\alpha) = \sum_{i=1}^{4} f_i'(\theta,\phi) \tag{4-34}$$

This function has the following characteristics:

1) It is s-convex because it is the sum of s-convex functions;

2) $$UB[\theta,\phi,\alpha] \geq V(\theta,\phi) \tag{4-35}$$

In other words,, since any g_o other than $g_o^*(\theta,\phi)$ would result in higher cost for $V(\theta,\phi)$, and also produce an α that is different from $\alpha^*(\theta,\phi)$, then any α used in $UB(\theta,\phi,\alpha)$ would make $UB(\theta,\phi,\alpha) \geq V(\theta,\phi)$. Therefore UB is an upper-bound for the objective function;

3) $UB[\theta,\phi,\alpha^*(\theta,\phi)] = V(\theta,\phi)$ for any (θ,ϕ) (4-36)

That is the equal sign of eq. (4-35) holds true when $\alpha = \alpha^*(\theta,\phi)$. Therefore

$$UB[\theta^*,\phi^*,\alpha^*(\theta^*,\phi^*)] = V(\theta^*,\phi^*) \tag{4-37}$$

4) $UB(\theta,\phi,\alpha) = V(\theta,\phi)$ at the boundary points where $d_1(\theta,\phi) = R[g_{max}(w_o),w_o]$ because, at those points, $WFL_{max}(d_1) = 0$. Thus the value of α does not matter.

How can we use this UB function to solve (P2)? Given a particular value of α, we can optimize the UB function to find the best (θ,ϕ) that corresponds to it. Now, if at some iteration we have the value of $\alpha^*(\theta^*,\phi^*)$ [without knowing (θ^*,ϕ^*)], and we optimize the UB function, then we will end up with (θ^*,ϕ^*). This is true since if (θ^*,ϕ^*) minimizes the true objective function, then it will also minimize $UB[\theta,\phi,\alpha^*(\theta^*,\phi^*)]$.

Solution Procedure of (P2) and Convergence Proof

1. Initialization:

   Let $k = 0$, and $\alpha^o = 1$. Start with a point (θ^o,ϕ^o) that satisfies (4-1), (4-2) and (4-3) (Selection of a good starting point is discussed on page 56).

2. Let $k = k + 1$. Find the point that will minimize $UB(\theta,\phi,\alpha^{k-1})$ s.t. (4-1), (4-2), and (4-3). Let the solution be (θ^k,ϕ^k).

3. For (θ^k,ϕ^k), solve (SP1) to find the best $g_o^*(\theta^k,\phi^k)$. Use equation (4-28) to find $\alpha^*(\theta^k,\phi^k)$.

4. If $\alpha^*(\theta^k,\phi^k) \neq \alpha^{k-1}$, we have an improvement in the objective function. Therefore, let $\alpha^k = \alpha^*(\theta^k,\phi^k)$ and go to step 2. Otherwise, the solution is (θ^k,ϕ^k), $g_o^*(\theta^k,\phi^k)$, and $h_o^*(\theta^k,\phi^k)$ is calculated from (4-12) and (4-13). Stop.

Next, a convergence proof of this procedure is given below.

We need to show that

$$\sum_{i=1}^{4} f_i(\theta^{k-1},\phi^{k-1}) \geq \sum_{i=1}^{4} f_i\ (\theta^k,\phi^k) \qquad (4\text{-}38)$$

Assume, at iteration k-1, that $\alpha^*(\theta^{k-1},\phi^{k-1}) \neq \alpha^{k-2}$. (Otherwise, we would have stopped there.) By setting $\alpha^{k-1} = \alpha^*(\theta^{k-1},\phi^{k-1})$, we have

$$\sum_{i=1}^{4} f_i(\theta^{k-1},\phi^{k-1}) = UB(\theta^{k-1},\phi^{k-1},\alpha^{k-1}) \qquad (4\text{-}39)$$

At iteration k, since (θ^k,ϕ^k) minimizes $UB(\theta,\phi,\alpha^{k-1})$, then

$$UB(\theta^{k-1},\phi^{k-1},\alpha^{k-1}) \geq UB(\theta^k,\phi^k,\alpha^{k-1}) \qquad (4\text{-}40)$$

But,

$$UB(\theta^k,\phi^k,\alpha^{k-1}) \geq \sum_{i=1}^{4} f_i(\theta^k,\phi^k) \qquad (4\text{-}51)$$

and inequality (4-38) follows.

Moreover,

$$\sum_{i=1}^{4} f_i(\theta^{k-1},\phi^{k-1}) > \sum_{i=1}^{4} f_i(\theta^k,\phi^k), \qquad (4\text{-}52)$$

if $UB(\theta^k,\phi^k,\alpha^{k-1}) > \sum_{i=1}^{4} f_i(\theta^k,\phi^k)$. (3-55)

The last inequality holds true if $\alpha^*(\theta^k,\phi^k) \neq \alpha^{k-1}$. Therefore, the true objective function improves from one iteration to another. Since the problem is s-convex, the solution procedure will converge to the global optimum solution [12]. Q.E.D.

Finding a Good Starting Point

Here, the following lower bound function is used to find a good starting point.

$$LB(\theta,\phi) = FN(w_o,d_1) + FN(w_o,d_2) + FN^*[h_{f,min}(\theta,\phi),d_3] + FN^*(0,d_3) \qquad (4\text{-}44)$$

where

$$\begin{aligned} h_{f,min}(\theta,\phi) &= FN(w_o,d_2) + FN^*(0,d_3) \\ &\quad - \{g_{max}(w_o) - FC\,[g_{max}(w_o),w_o,d_1]\} \\ &= FN(w_o,d_2) + FN^*(0,d_3) - WFL_{max}(d_1) \qquad (4\text{-}45) \end{aligned}$$

This lower bound function is s-convex, and $LB(\theta,\phi) \leq V(\theta,\phi)$. A physical explanation for $LB(\theta,\phi) \leq V(\theta,\phi)$ follows:
Each term in the LB function represents the minimum amount of fuel the various A/C need for each particular leg regardless of the actual fuel consumed or needed for the other legs. That is, for each leg of the flight we assume the other A/C will come to the refueling point with the maximum amount of fuel possible. The equality holds at the boundary points where $d_1(\theta,\phi) = R[g_{max}(w_o),w_o]$. How can the LB function be used? Start with any point inside the region described by the

intersection of inequalities (4-1), (4-2) and (4-3), and solve the following problem:

$$\begin{array}{ll} \text{Min} & LB(\theta,\phi) \\ \text{S.T.} & (4\text{-}1),\ (4\text{-}2) \text{ and } (4\text{-}3) \end{array}$$

If the optimal solution to $LB(\theta,\phi)$ lies on the boundary where $d_1 = R[g_{max}(w_o,w_o]$, then it is the optimal solution to (P2), and we do not have to search any further. If the minimum value of the lower bound function is $> H_{max} + g_{max}(w_o)$, then there is no feasible solution to this problem.

There is no guarantee that the resulting point will be feasible if

$$\underset{(\theta,\phi)}{\text{Min}} \ \{LB(\theta,\phi)\} \leq H_{max} + g_{max}(w_o);$$

however, its feasibility could be checked using inequality (4-10). If it is feasible, then all subsequent points will be feasible also. (See the first remark about the objective function on page 45.) Even if it is not feasible, this will not render the solution procedure of (P2) useless because the feasibility of the starting point is not necessary as long as the point satisfies inequalities (4-1), (4-2) and (4-3). The procedure would still be valid because the objective function improves from one iteration to another. Improvement and convergence do not depend on the point satisfying inequality (4-10). In fact, we need to check the feasibility of the point only at the end when the procedure stops and convergence is reached. We then check to see if the final point satisfies inequality (4-10). If it does, then the problem has a feasible solution and the solution at hand is the optimum feasible solution we are seeking. Otherwise the problem does not have a feasible solution at all.

Computational Results

The above mentioned procedure was programmed on a Vax 11/750 using BASIC. There was no attempt to make it efficient. Several test problems were solved to represent a variety of possible geographic configuration and complexities. The minimum CPU time was approximately 0.5 second (2 iterations) and the maximum was almost 1.0 second (4 iterations) for convergence within one degree longitude and latitude. Table 4-2 shows these test problems with their results, while Table 4-1 provides the key to understand the entries of Table 4-2.

TABLE 4-1: Key for trial runs chart of Table 4-2

Location	Longitude	Latitude	Code
New Jersey	75W	40N	A
Delaware	75W	38N	B
North Carolina	78W	35N	C
Puerto Rico	66W	18N	D
Azores Islands	25W	37N	E
Iceland	20W	65N	F
Germany	10E	50N	G
Turkey	30E	40N	H
Saudi-Arabia	47E	25N	I
Egypt	28E	30N	J
England	0E/W	52N	K

W = Cargo Weight	Code
100,000 lbs	1
200,000 lbs	2

Fuel ACCR = Fuel Accuracy

100 means to within 100 lbs
10 means to within 10 lbs

POS ACCR = Position Accuracy

1 means to within 1 degree
2 means to within 1 minute
3 means to within 1 second

TABLE 4-2: Trial runs for Problem (P2)

Run No.	"O"	"D"	"B"	Initial Refueling Point (Guess)	W	Fuel ACCR	Pos ACCR	g_o (lbs)	Fuel Transferred to Transport (lbs)	Fuel Consumed By Tanker (lbs)	Total Fuel (lbs)	Optimal Location	Time (Sec)	Itera-tions
1	A	H	D	40W 35N	2	10	1	113,521	136,704	149,055	399,280	49W 40N	0.6	2
2	G	C	F	35W 40N	2	10	1	79,431	130,541	1,058	211,030	20W 65N	1.0	4
3	B	I	E	30W 30N	1	10	1	137,142	144,464	5,866	287,472	25W 37N	0.8	2
4	B	J	D	30W 25N	2	10	1	118,712	152,613	156,783	428,008	42W 36N	0.5	2
5	C	K	F	30W 50N	2	10	1	129,882	54,425	19,269	203,576	29W 63N	0.6	2

"O" = Origin

"D" = Destination

"B" = Tanker Base

g_o = Initial fuel for transport A/C

h_o = Initial fuel for tanker A/C

= Fuel consumed by tanker + fuel transferred to transport

CHAPTER V

PROBLEM (P3)--MODEL AND SOLUTION

Two A/C
1 Refueling Required
(2 Legs)

Formulation of (P3)

Consider the case where there are two transport A/C available at the origin base "0," and two tanker A/C at another base ready to refuel the transport A/C if necessary. It is desired to find out how to divide the cargo weight optimally among both transport A/C. Note that the chosen value of w_i might necessitate an aerial refueling, in which case the optimal refueling point also needs to be determined. Also, it may not be necessary to use both A/C. Rather, we may use only one A/C if feasible. The chosen option will be the one which results in the lowest total cost. To find this we need to solve the following problem:

$$\text{(P3)} \qquad TC2(W) = \text{Min } TC(w_1,w_2) = \text{Min } [C(w_1) + C(w_2)]$$

$$\text{S.T.} \qquad w_1 + w_2 \leq W$$

$$w_1 \geq 0, \; w_2 \geq 0$$

where, TC2(W) = Total fuel cost incurred by transporting cargo which weighs W when there are only two A/C available, and C(w) = cost of transporting a cargo weight of w by one A/C.

$$\text{i.e.} \quad C(w) = \begin{cases} 0 & \text{if } w = 0 \\ C_o(w) = FN\ (w, D_{OD}) & \text{if } 0 < w \leqslant Z_o \\ C_1(w) = \text{solution of of (P2)} & \text{if } Z_o < w \leqslant Z_1 \\ \infty & \text{if } Z_1 < w \end{cases}$$

$TC(w_1, w_2) = C(w_1) + C(w_2)$, where

Z_o = the maximum cargo weight one transport A/C can carry from "0" to "D" without refueling.

Z_1 = the maximum cargo weight one transport A/C can carry from "0" to "D" by refueling once. ($Z_1 > Z_o$.)

At the end of this chapter we present simple ways to compute Z_o and Z_1. Figure 5-1 shows a typical C(w) function. This function is piecewise convex and strictly increasing. The jump at Z_o depends on D_{OD}, W and the location of "B."

Solution of (P3)

To make the solution of (P3) easier, we can rewrite it in terms of one variable instead of two, i.e.,

$$\text{(P3)} \qquad TC2(W) = \text{Min } TC(w_1,\ W - w_1) = \text{Min } C(w_1) + C(W - w_1)$$

$$\text{S.T.} \qquad 0 \leqslant w_1 \leqslant W$$

Solving this problem allows us to identify seven (7) cases, depending upon the value of W relative to Z_o and Z_1. These cases have simple solutions. The analysis of these seven cases is presented next, followed by a summary of a simple solution procedure.

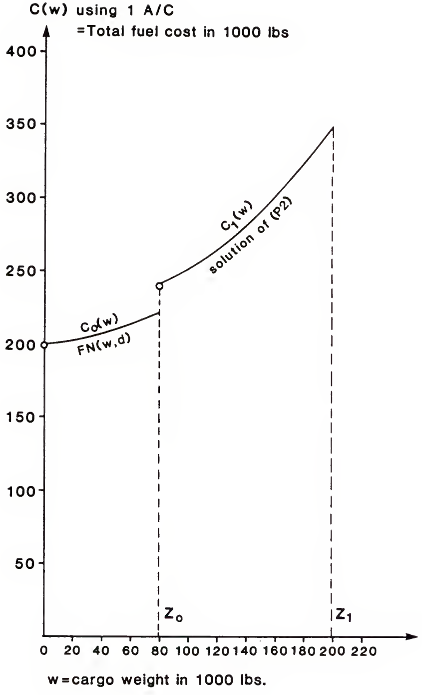


Figure 5-1: Cost when refueling is used.

Case 1: (1D, 2D) $W \leq Z_o$

Here, one A/C can carry the whole cargo directly to the destination without refueling. The cost is $C_o(W) = FN(W, D_{OD})$. Also, this implies that two A/C can carry the cargo without refueling. From problem (P1), theorem (2-1), we know that it is best to divide W equally among both A/C. The cost will be $2\, C_o\, (\frac{W}{2}) = 2\, FN(\frac{W}{2}, D_{OD})$. This has to be compared with $C_o(W)$, and the option with the least cost should be chosen.

Case 2: (1R, 2D) $Z_o < W \leq \text{Min}\, \{2\, Z_o, Z_1\}$

Again two A/C can carry the whole cargo directly to the destination, but aerial refueling is needed if only one A/C is used. To find out how much the one A/C scenario costs, we need to solve problem (P2) for $w_o = W$ and also find the optimal refueling point. The cost is $C_1(W) = V(\theta, \phi)$. Again costs for both scenarios have to be compared. Figure (5-2) shows TC $(w_1, W - w_1)$ as a function of w_1 for this case.

Case 3: (2D) $Z_1 < W \leq 2\, Z_o$

Here one A/C can not transport the whole cargo (W) even if aerial refueling is used. But, two A/C can transport the cargo directly. The cost is $2\, C_o(\frac{W}{2})$, and this is the only option we have.

Case 4: (1R, 2R, 1D1R) $2Z_o < W \leq Z_1$

Here, two A/C cannot carry the whole cargo (W) directly to "D." At least one of them has to refuel. If only one A/C refuels, then the optimum weights are $w_1^* = Z_o$ and $w_2^* = W - Z_o$. The cost is $C_o(Z_o) + C_1(W - Z_o)$. Therefore, problem (P2) has to be solved for $w_o = W - Z_o$ to find $C_1(W - Z_o)$ and the optimum refueling point. But if both A/C

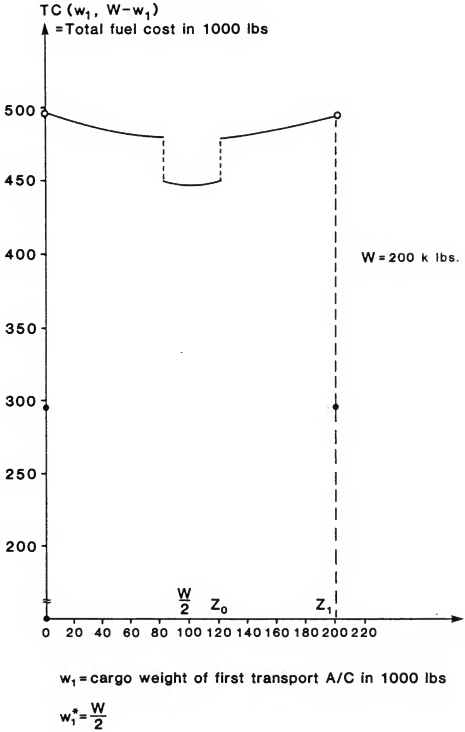


Figure 5-2: Cost for Case (2).

refuel then the optimal solution is $w_1^* = w_2^* = W/2$, and the total cost is $2\ C_1(W/2)$. Thus, we have to solve problem (P2) for $w_o = W/2$ to find the optimal refueling point and the value of $C_1(W/2)$. Also, since $W < Z_1$, then one A/C would be able to carry the whole cargo when aerial refueling is used. Again, we need to solve problem (P2) for $w_o = W$ to find $C_1(W)$ and the optimal refueling point. Finally, all three costs have to be compared and the least expensive option is chosen. Figs. (5-3) and (5-4) show a typical $TC(w_1, W - w_1)$ curve as a function of w_1 for this case.

Case 5: (2R, 1D1R)

$$2Z_o < W < Z_o + Z_1 \text{ and } Z_1 < W \rightarrow \text{Max}\{2Z_o, Z_1\} < W < Z_o + Z_1$$

Again, two A/C have to be used, and at least one of them must refuel. The "two A/C refueling" option costs $2\ C_1(W/2)$ which is obtained by solving problem (P2) for $w_o = W/2$. The other option, where one A/C refuels and the other flies directly, costs $C_o(Z_o) + C_1(W - Z_o)$. The cost $C_1(W - Z_o)$ is found by solving (P2) for $w_o = W - Z_o$. Again, both costs have to be compared.

Case 6: (2R) $\qquad Z_o + Z_1 < W < 2Z_1$

Here, the two A/C have to be used, and both have to refuel. This is the only option. Thus, the cost is $2\ C_1(W/2)$, and (P2) has to be solved for $w_o = W/2$ to find this cost.

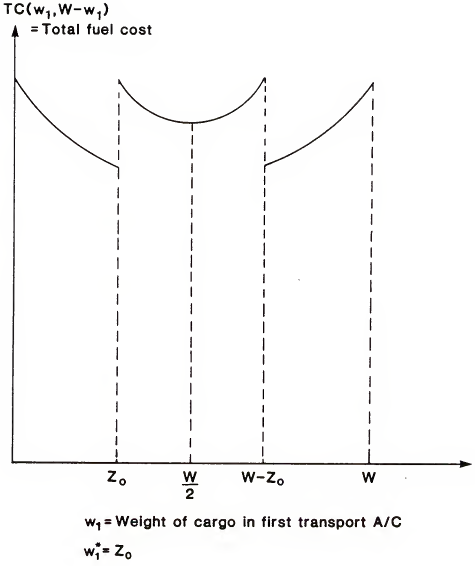


Figure 5-3: Cost for case (4) $2Z_o < W < Z_1$.

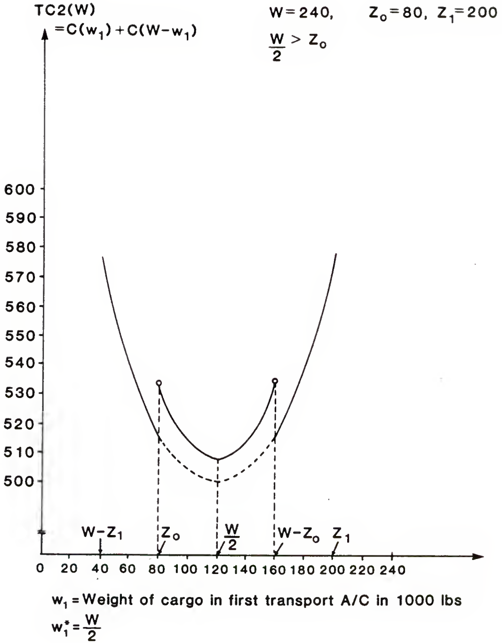


Figure 5-4: Another example for the cost of case (4).

Case 7: (infeasible problem) $2Z_1 < W$

Here, two A/C are not enough to carry the whole cargo even if they are both fully loaded and mid-air refueling is used. Therefore, more A/C are needed, or aerial refueling might be used twice.

Before we present the summary of the solution procedure, we now offer a simple way to find Z_o and Z_1.

Finding the Value of Z_o

1. Solve the following equation for w: $MTOW - EW - w = FN\ (w, D_{OD})$, where $FN\ (w, D_{OD})$ is defined in eq. (2-20). Call this value w_a.
2. Solve the next equation for w: $F_{max} = FN\ (w, D_{OD})$. Call this value w_b.
3. Set $Z_o = \text{Min}\ \{w_a, w_b, w_{max}\}$ where w_{max} = the maximum weight that can be carried inside the A/C due to structural weight limitations.

   This procedure is based on the fact that Z_o must satisfy the following conditions:

   $Z_o \leq w_{max}$,

   $FN\ (Z_o, D_{OD}) = g_{max}(Z_o)$, where

   $g_{max}(Z_o) = \text{Min}\ \{F_{max}, MTOW - EW - Z_o\}$.

Finding the Value of Z_1

The real issue is determining if $w_o \leq Z_1$, for any value of w_o. That is we need to see if $W \leq Z_1$, $W/2 \leq Z_1$ or $W - Z_o \leq Z_1$ (or any other value of w_o for that matter). This is accomplished by solving problem (P2) for that particular value of w_o and determining if $V(\theta,\phi) \leq H_{max} + g_{max}\ (w_o)$. If this relation holds true, then the problem is feasible

and $w_o < Z_1$. As a by-product we get the value of the total fuel consumption $C_1(w_o)$ which is equal to $V(\theta, \phi)$, together with g_o^*, h_o^* and (θ^*, ϕ^*). Note that Z_1 cannot exceed the value of w_{max}, thus we do not need to solve (P2) unless $w_o < w_{max}$. In other words, we can say that $w_o > Z_1$ if $w_o > w_{max}$.

Summary of Solution Procedure of (P3)

1. Find Z_o.
2. If $W > Z_o$ go to step (3). Otherwise compare $C_o(W)$ with 2 $C_o(W/2)$. If $C_o(W) < 2\ C_o\ (W/2)$, use one A/C flying directly (1D). Its initial fuel is $C_o(W)$. Otherwise use two A/C (2D), each having initial fuel equals to $C_o(W/2)$. Stop.
3. If $W > 2\ Z_o$, go to step 5. Otherwise, solve (P2) for $w_o = W$ to see if $W < Z_1$. If $V(\theta,\phi) > H_{max} + g_{max}\ (W)$, go to step 4 because $W > Z_1$ (That is (P2) with $w_o = W$ is infeasible). Otherwise, compare $C_1(W) = V(\theta,\phi)$ with 2 $C_o(W/2)$. If $C_1(W) < 2\ C_o(W/2)$, use one A/C (1R). Get the values of g_o^*, h_o^* and (θ^*, ϕ^*) from the solution of (P2). Otherwise use two A/C flying directly (2D), where $g_1^* = g_2^* = C_o(W/2)$. Stop.
4. Use two A/C flying directly (2D). $g_1^* = g_2^* = C_o(W/2)$. Stop.
5. Solve (P2) with $w_o = W/2$ to see if $\frac{W}{2} < Z_1$. If (P2) is feasible go to step 6. Otherwise, we have $W > 2\ Z_1$ which means two A/C cannot carry the whole cargo even if aerial refueling is used. Therefore, we need more A/C or need to refuel twice, and problem (P3) is infeasible. Stop.
6. Solve (P2), again, but for $w_o = W - Z_o$ to see if $w < Z_o + Z_1$ (or $W - Z_o < Z_1$). If the new (P2) is feasible go to step 7. Otherwise, we

use two A/C, both refueling (2R). $w_1^* = w_2^* = W/2$. g_o^*, h_o^* and (θ^*, ϕ^*) are obtained from the solution of the first (P2) problem mentioned in step 5. Stop.

7. Solve (P2), this time, for $w_o = W$ to see if $W \leq Z_1$. If the recent (P2) problem is feasible, go to step 8. Otherwise, compare the cost of step 5 and step 6, i.e., compare 2 $C_1(W/2)$ with $C_o(Z_o) + C_1(W2 - Z_o)$. If $2C_1(W/2)$ is lower, we have $w_1^* = w_2^* = W/2$ and the two A/C will need to refuel (2R). Get the values of g_o^*, h_o^* and (θ^*, ϕ^*) for both A/C from the solution obtained in step 5. Otherwise, we have $w_1^* = Z_o$ and $w_2^* = W - Z_o$ (1D1R). Get the values of g_o^*, h_o^* and (θ^*, ϕ^*) for the second A/C from the solution obtained in step 6. Stop.

8. Here, we have the following three options:

   (a) $w_1^* = W$, $w_2^* = 0$, with cost = $C_1(W)$ obtained in step 7. (1R)

   (b) $w_1^* = Z_o$, $w_2^* = W - Z_o$, with cost = $C_o(Z_o) + C_1(W - Z_o)$ obtained in step 6. (1D1R)

   (c) $w_1^* = w_2^* = W/2$, with cost = 2 $C_1(W/2)$ obtained in step 5. (2R).

   Compare the cost and pick the lowest one. Get the values of g_o^*, h_o^*, (θ^*, ϕ^*) from the solution obtained in the equivalent step. Stop.

Figure 5-5 gives a graphical summary of the solution procedure, while Fig. 5-6 shows how the fuel cost varies with W, the total cargo weight, for a C-5 A. Also it shows how W will be transported.

Before we end this chapter, we would like to give an example to illustrate how simple the aforementioned solution procedure is.

Example 1:

Suppose we have 2 (identical) A/C, and that Z_o = 80 k lbs and Z_1 = 180 k lbs (known). What would the solution be for different values of

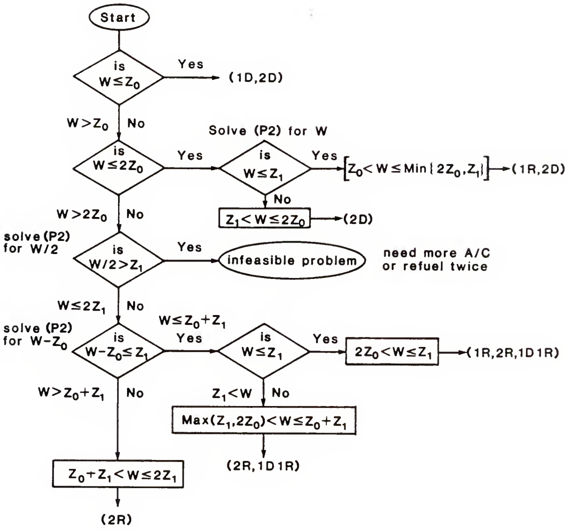


Figure 5-5: Graphical summary of the solution procedure of (P2).

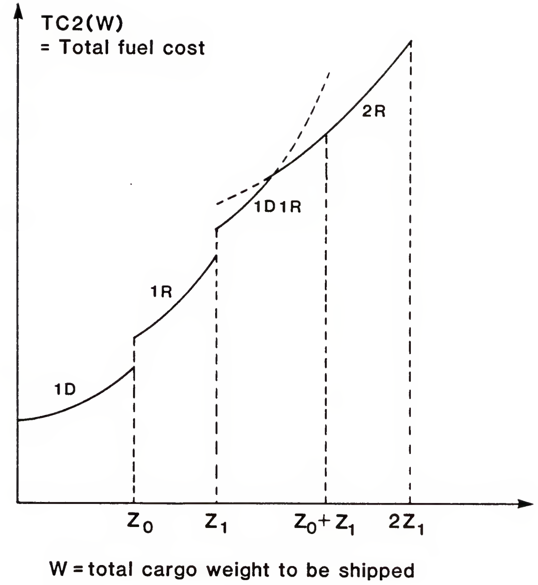


Figure 5-6: Fuel cost as a function of W for a typical C5-A.

W? Specifically, what is the solution of W = 230 k lbs.? What happens if we do not know the value of Z_1?

Solution:

I. Table 5-1 shows the solution when W lies in any interval.

II. For the particular value of 230 k lbs for W, we have two options to choose from:

a) To have 2 refueling A/C with $w_1 = w_2 = W/2 = 115$ k lbs. and total fuel consumption = $TC_a = 2C_1(115)$. The value of $C_1(115)$ is found by solving (P2) for $w_o = 115$.

b) To have one A/C fly directly to the destination with $w_1 = Z_o = 80$ k lbs., and the other to have $w_2 = W - Z_o = 230 - 80 = 150$ k lbs. The fuel consumption for this option = $TC_b = C_o(80) + C_1(150)$. The value of $C_1(150)$ is found by solving (P2) for $w_o = 150$. Finally, we compare TC_a and TC_b to choose the least cost option.

III. Now, suppose we are not given the value of Z_1 in advance. We have to proceed in the following way as suggested by Fig. 5-5. Note that $W = 230 > 2Z_o = 160$. Now, we have to solve (P2) for $W/2 = 115$ to see if $W/2 \leq Z_1$. When we solve this problem, we see it is feasible and we also get the value of $C_1(W/2 = 115)$. Next, we check if $W - Z_o = 230 - 80 = 150 \leq Z_1$ by solving (P2) for $W - Z_o = 150$. We find out that it is, and we also get the value of $C_1(W - Z_o = 150)$. Finally we check if $W = 230 \leq Z_1$ by solving (P2) for $W = 230$. We find that it is not. This means we have:

$$\text{Max}\{Z_1, 2Z_o\} < W \leq Z_o + Z_1 .$$

TABLE 5-1: Solution of example 1

Interval for W (k lbs)	0 to 80	80.001 to 160	160.001 to 180	180.001 to 260	260.001 to 360	above 360
Solution(s)	1D or 2D	1R or 2D	1R or 2R or 1D1R	2R or 1D1R	2R	Need more A/C or refuel twice

The solution would be the best of the two following options:

option a (2R) with cost = $2C_1(115)$

and option b (1D1R) with cost = $C_1(80) + C_1(150)$

as before.

CHAPTER VI

PROBLEM (P4)--MODEL AND SOLUTION

N A/C
1 Refueling required
(2 Legs)

Problem Description

Consider the situation where there are N transport A/C at the origin base "O." Some or all of these A/C can be used to transport cargo weighing W to the destination base "D." It is desired to know how many of these A/C to use and how to divide the cargo weight among them optimally in order to minimize total fuel consumption. If the load of some of these A/C exceeds Z_o, then aerial refueling is necessary and we also have to find the optimal refueling point.

Solution of (P4)

The solution procedure is an extension of the procedure used to solve Problem (P3). First, we have to find the different feasible ways in which we can transport the cargo. Then, compare their costs and choose the one that has the lowest total cost. To find these combinations we first find the values of M_{min} and Q which are defined as

M_{min} = the minimum number of transport A/C needed to transport the cargo directly without aerial refueling.

Q = the minimum number of transport A/C needed to transport the cargo when aerial refueling is used for all of them.

Next we show how we can find M_{min} and Q. Also, we will give an example to illustrate the procedure of finding the different feasible solutions. Finally, a summary of the solution procedure is given.

Finding M_{min}

$$M_{min} = \left\lceil \frac{W}{Z_o} \right\rceil \quad \text{where} \tag{6-1}$$

$\lceil X \rceil$ = the smallest integer greater than or equal to X

Finding Q

$$Q = \left\lceil \frac{W}{Z_1} \right\rceil \tag{6-2}$$

If Z_1 is not known then we can find Q as follows:

1. Since $Z_1 < w_{max}$, let $q = \left\lceil \frac{W}{w_{max}} \right\rceil$.
2. Solve problem (P2) with $w_o = \frac{W}{q}$ to see if it is feasible.
3. If it is feasible, then set Q = q and stop. Otherwise make q = q + 1 and go to step (2).

Example

Suppose we have: $W = 490$, $Z_o = 120$, $Z_1 = 180$ and $w_{max} = 200$. Find the different feasible solutions

$$M_{min} = \left\lceil \frac{490}{120} \right\rceil = 5$$

$$Q = \left\lceil \frac{490}{180} \right\rceil = 3$$

(even if we do not know Z_1, we can find Q as suggested before, i.e.,

Since $q = \lceil \frac{W}{w_{max}} \rceil = 3$, solving problem (P2) with $w_o = \frac{W}{q} = 163.33$, we find that q is feasible. Thus Q = q = 3.)

So far we have two feasible solutions, namely (5D) and (3R), i.e., five A/C can carry the whole cargo directly or three A/C can carry it but they need to refuel in mid-air. Now let us see if [1D 2 R] is feasible. We get this combination from (3R) by reducing the number of refueling A/C by one, and letting one A/C fly directly. Since we can have up to Z_o in the A/C that is flying directly, we then check to see if (Q-1) refueling A/C can carry ($W-Z_o$) weight. That is, we check if two refueling A/C can carry 370 k lbs. To do this we see if

$$\frac{W - Z_o}{Q - 1} < Z_1 ,$$

(i.e., check whether $\frac{370}{2} < 180$). Since it is not, we check whether [2D (Q - 1) R] is feasible, i.e., check whether (2D 2R) is feasible. Again, this translates to determining if

$$\frac{W - 2Z_o}{Q - 1} < Z_1,$$

(i.e., whether $\frac{250}{2} < 180$.) Since it is, we now have a new feasible solution which is (2D 2R). Starting with this new solution, we try the same process of reducing the number of refueling A/C by one and increasing the number of A/C that are flying directly by one and checking its feasibility. If not feasible, then again increase the number of A/C that are flying directly. Thus, we check whether (3D 1R) is feasible by determining if

$$\frac{W - 3Z_o}{Q - 2} < Z_1 \text{ or in our example if } \frac{130}{1} < 180?$$

Since this is true, we have another new solution, namely (3D 1R). Now we repeat this to see if (4D) is feasible. But we know that $M_{min} = 5$ implies that (4D) is not feasible and therefore the feasible solutions are: (5D), (3D 1R), (2D 2R), (3R).

Now we come to the problem of how to divide the cargo among the A/C to be used. For the solution (5D) and (3R) [or equivalently (M_{min} D) and (QR)], we divide the cargo weight equally among the A/C. This is because in both cases the total cost function is continuous, increasing and convex in their respective region. This is also shown to be optimal in problems (P1) and (P2). Now we come to the solution (3D 1R) and (2D 2R). We saw from the solution of problem (P3) that whenever we have a combination of A/C of which some fly directly to the destination while the rest refuel, the cargo load for the A/C flying directly should be set at maximum. That is, $w_i^* = Z_o$ for A/C flying directly while the rest of the cargo weight is divided equally among the refueling A/C. In conclusion, the different solutions for the example problem are as follows:

1. Have five A/C fly directly to the destination, each having a load equal to $w_i^* = \frac{W}{M_{min}} = \frac{490}{5} = 98$ k lbs. Total fuel cost = 5 C_o (98 k).
2. Have three A/C refuel, each having a load equal to $w_i^* = \frac{W}{Q} = \frac{490}{3} =$ 163.3 k lbs. The cost will be 3 C_1 (163.3 k lbs.). To find this, we have to solve problem (P2) for $w_o = 163.3$ k lbs.
3. Have three A/C fly directly with each carrying $w_i^* = Z_o = 120$ k lbs ($i = 1,2,3$), and a fourth A/C refuel in mid-air, carrying $w_4^* =$ $(W-3Z_o)/1 = W - 3Z_o = 130$ k lbs. The total cost is = 3 C_o (Z_o)

$+ C_1 (W - 3Z_o) = 3 C_o (120) + C_1 (130)$. The quantity C_1 (130) can be found by solving problem (P2) for w_o = 130 k lbs.

4. Have two A/C fly directly with each carrying $w_i^* = Z_o$ = 120 k lbs (i = 1,2), and another two refuel in mid-air, each carrying $w_i = (W-2Z_o)/2$ = 125 k lbs. (i = 3,4). The total cost is $= 2 C_o (Z_o) + 2C_1 [(W-2Z_o)/2] = 2 C_o (120) + 2 C_1 (125)$. The quantity C_1 (125) can be found by solving problem (P2) for w_o = 125. Finally, compare the four different total costs and select the minimum one.

Summary of the Solution Procedure of Problem (P4)

1. Find M_{min} according to eq. (6-1).
2. Find Q according to eq. (6-2) or use the suggested procedure (found on page 72) if Z_1 is not known.
3. Set $\ell_D = 0$, $\ell_R = Q$, where

   ℓ_D = the number of A/C flying directly to the destination, and

   ℓ_R = the number of A/C that need to refuel in mid-air

4. Let $\ell_D = \ell_D + 1$, $\ell_R = \ell_R - 1$
5. If $\ell_R = 0$ go to step 8.
6. Check the feasibility of the current ($\ell_D D\ \ell_R R$) solution as follows:

   If $\dfrac{W - \ell_D Z_o}{\ell_R} < Z_1$ then this solution is feasible. Record this solution and go to step 4. Otherwise go to step 7.

7. Increase ℓ_D by 1, i.e. let $\ell_D = \ell_D + 1$. If $\ell_D = M_{min}$, go to step 8. Otherwise go to step 6.

8. Compute the cost of each feasible solution as follows:

(a) For the ($M_{min}D$) solution we have:

$$w_i^* = \frac{W}{M_{min}} \qquad i = 1,2, \ldots M_{min}$$

$$\text{Total cost} = M_{min} * C_o \left(\frac{W}{M_{min}}\right)$$

(b) For the (QR) solution we have

$$w_i^* = \frac{W}{Q} \qquad i = 1,2, \ldots Q$$

$$\text{Total cost} = Q * C_1 \left(\frac{W}{Q}\right)$$

Solve problem (P2) with $w_o = \frac{W}{Q}$ to find $C_1 \left(\frac{W}{Q}\right)$

(c) For any solution of the form ($\ell_D D \; \ell_R R$) we have:

$$w_i^* = Z_o \qquad \text{for } i = 1,2, \ldots \ell_D ,$$

$$w_i^* = \frac{W - \ell_D Z_o}{\ell_R} \qquad \text{for } i = \ell_D + 1, \ldots \ell_D + \ell_R .$$

$$\text{Total cost} = \ell_D * C_o (Z_o) + \ell_R * C_1 \left(\frac{W - \ell_D Z_o}{\ell_R}\right) .$$

Solve problem (P2) with $w_o = \frac{W - \ell_D Z_o}{\ell_R}$ to find $C_1 \left(\frac{W - \ell_D Z_o}{\ell_R}\right)$.

9. Compare the different costs and pick the minimum one.

Figure 6-1 gives a graphical summary of the solution procedure.

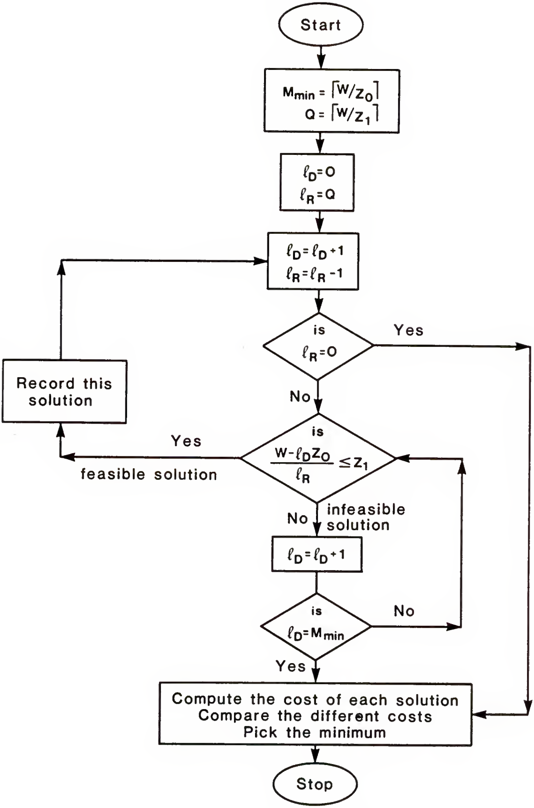


Figure 6-1: Graphical summary of the solution procedure of problem (P4)

CHAPTER VII

PROBLEM (P5)--MODEL AND SOLUTION

1 A/C
2 Refueling Required
(3 legs)

Problem Description

This chapter deals with the situation where the transport A/C needs to refuel twice. First we will describe the problem and indicate when this need arises. Second, we will provide an iterative solution procedure for problem (P5), which depends on the solution of problem (P2).

In general, there are two cases where the transport A/C needs to refuel twice. The first case involves only one tanker base which has two tanker A/C. We know from problem (P2) that the feasible points must lie in a region whose points must satisfy eqs. (4-1), (4-2) and (4-3). However, it could be the case that these equations do not simultaneously define one region. Rather, two regions are formed: one region where eqs. (4-1) and (4-3) are satisfied, and another where eqs. (4-2) and (4-3) are satisfied. (See Figures (7-1) and (1-2).) Therefore, one tanker can fly to a point in the first region to meet and refuel the transport A/C. It goes back to its base, while the transport flies into the second region. There, the transport meets another tanker from the same base to refuel for the second time. Afterwards, it heads directly for the desired destination. Here, aerial refueling is done twice, using

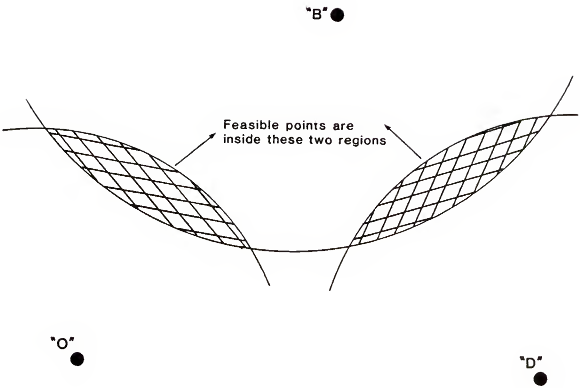


Figure 7-1: One case where refueling twice is necessary.

tankers originating from the same base. We need to optimally locate the two refueling points and specify the initial fuel for the transport and each tanker.

The other case involves two tanker bases and is very similar to the first one. (See Figs. (1-1) and (7-2).) There, the tankers originate from two different bases. In our treatment here, we will deal with the more general case, namely the second one where we have two tanker bases.

Motivation for the Solution Procedure of (P5)

We will solve (P5) in an iterative way where each iteration consists of two steps (2 subproblems). At each step, we fix one refueling point and solve for the other. Then we use the results of the first subproblem, as an input to the second, to resolve for the original refueling point. This process is repeated and if convergence can be reached then the solution is optimal. Note that each subproblem is essentially problem (P2) with minor changes.

Let "r_1" = the first refueling point

"r_2" = the second refueling point

d_1 = distance between "0" and "r_1"

d_2 = the distance between "r_1" and "r_2"

d_3 = the distance between "r_1" and "B1"

d_1' = d_2

d_2' = the distance between "r_2" and "D"

and d_3' = the distance between "r_2" adn "B2"

(See Fig. (7-3))

Now we can identify the following two subproblems:

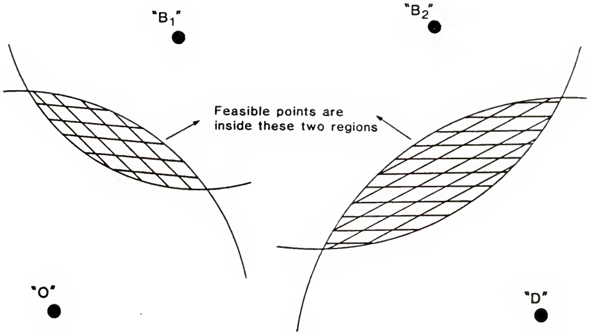


Figure 7-2: Another case where refueling twice is necessary.

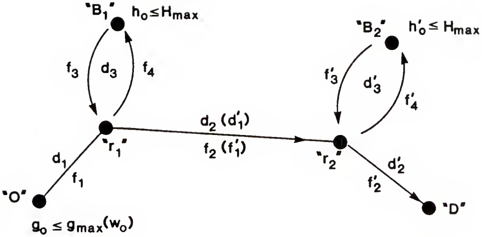


Figure 7-3: Problem (P5) where refueling is done twice.

Subproblem I

Suppose we are given "r_1". Let g_I = the maximum amount of fuel the transport A/C can have immediately after it refuels for the first time.

Then,
$$g_I = g_{max}(w_o) + H_{max} - Fc[g_{max}(w_o), w_o, d_1] - FC^*(H_{max}, 0, d_3) - FN^*(0, d_3) \qquad (7\text{-}1)$$

Letting "r_1" be our new origin base, solve for "r_2", g'_o and h'_o. This translates to:

$$\text{(SP5I)} \qquad \text{Minimize} \sum_{i=1}^{4} f'_i$$

$$\begin{aligned}
\text{(S.T.)}\ f'_1 &= FC\,(g'_o, w_o, d'_1) \\
f'_2 &= FN\,(w_o, d'_2) \\
f'_3 &= FC^*\,(h'_o, 0, d'_3) \\
f'_4 &= FN^*\,(0, d'_3) \\
0 &< g'_o < g_I \\
0 &< h'_o < H_{max} \\
& g'_o + h'_o > \sum_{i=1}^{4} f'_i \\
d'_1 &< R\,[g_I, w_o] \\
d'_2 &< R\,[G_{max}(w_o), w_o] \\
d'_3 &< \tfrac{1}{2}\, R^*\,(H_{max}, 0)
\end{aligned}$$

(See Fig. (7-4))

Subproblem II

Suppose we are given "r_2". Let g_f = the amount of fuel the transport A/C must have when it reaches "r_2" in order for it to be able to reach "D" after it refuels at "r_2" (g_f could be equal to zero). That is

$$g_f = \text{Max}\,\{FN(w_o, d'_2) + FC^*(H_{max}, 0, d'_3) + FN^*(0, d'_3) - H_{max}\,,\ 0\} \qquad (7\text{-}2)$$

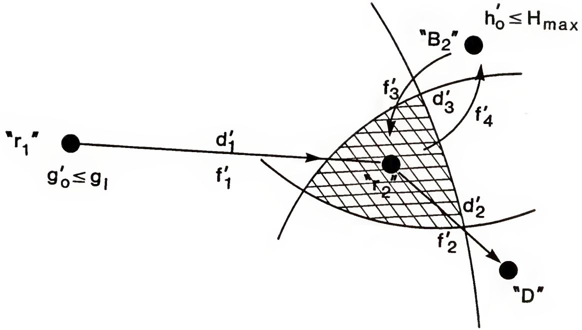


Figure 7-4: Subproblem I--solving for "r_2", given "r_1".

Now treating "r_2" as our destination and requiring that the transport A/C ends up with an amount of fuel equals to g_f, solve for "r_1," g_o and h_o. This translates to

$$\text{(SP5II)} \qquad \text{Minimize} \quad \sum_{i=1}^{4} f_i$$

$$\begin{aligned}
\text{(S.T.)}\ f_1 &= FC\ (g_o, w_o, d_1) \\
f_2 &= FN\ (w_o + g_f,\ d_2) \\
f_3 &= FC^*\ (h_o,\ 0,\ d_3) \\
f_4 &= FN^*\ (0,\ d_3) \\
0 &\leq g_o \leq g_{max}(w_o) \\
0 &\leq h_o \leq H_{max} \\
g_o + h_o &\geq \sum_{i=1}^{4} f_i + g_f \\
d_1 &\leq R\ [g_{max}(w_o),\ w_o] \\
d_2 &\leq R[G_{max}(w_o + g_f),\ w_o + g_f] \\
d_3 &\leq \frac{1}{2} R^*\ [H_{max},\ 0]
\end{aligned}$$

(See Fig. (7-5))

<u>Summary of the Solution Procedure of Problem (P5)</u>

1. Let $k = 0$. Choose a very small number $\varepsilon > 0$ as tolerance for the stopping criterion.
2. Start with a point that satisfies eqs. (4-1) and (4-3). Call it r_1^o.
3. Let $k = 1$.
4. Compute g_I according to eqs. (7-1).
5. Using r_1^{k-1}, solve subproblem (SP5I) to find r_2^k, g_o' and h_o'.
6. Using r_2^k, compute g_f according to eq. (7-2).
7. Using r_2^k, solve subproblem (SP5II) to find r_1^k, g_o, and h_o.

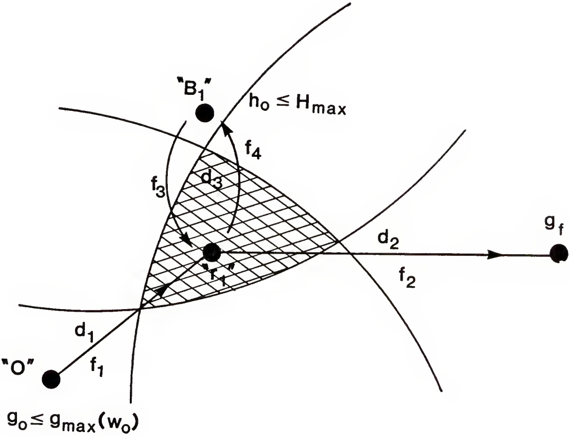


Figure 7-5: Subproblem II--solving for "r_1", given "r_2".

8. If the distance between r_1^k and $r_1^{k-1} > \varepsilon$, let $k = k + 1$ and go to step (4). Otherwise, convergence is reached and the solution is r_1^k, r_2^k, g_o, h_o, g'_o and h'_o.

What can we say about the solution procedure of problem (P5)? Does it converge? If it does, is the solution a global optimal solution? If we can not show that, can we at least show that the solution is stable? That is, if the process happens to reach the global optimal solution, will the procedure stop there or will it drift away from it? The following section gives us some facts about the solution procedure of problem (P5) which will answer the questions we have just raised.

Facts About Problem (P5)

1) The solution procedure of each subproblem, namely (SP5I) and (SP5II), converges to the optimal solution.

   Proof:

   See the proof of the solution procedure of problem (P2) because each subproblem is essentially problem (P2).

   As a consequence of this fact, and in order to help us talk about the convergence of the solution procedure of (P5), we restate steps 5 and 7 of the solution procedure as follows:

2) a- Given point r_1^{k-1}, then point r_2^k is the best refueling point corresponding to it; thus it is the optimal solution to (SP5I)

   b- Given r_2^k, then point r_1^k is the best refueling point corresponding to it; thus it is the optimal solution to (SP5II).

3) If $r_1^k \neq r_1^{k-1}$, then we have an improvement.

Proof:

We need to show that the total fuel consumption at iteration k is at most equal to that at iteration k-1. In mathematical terms, we need to show that

$$h_o^k + g_o^k + h_o'^k + g_o'^k \leq h_o^{k-1} + g_o^{k-1} + h_o'^{k-1} + g_o'^{k-1} \qquad (7-3)$$

Given r_1^{k-1} and the fact that r_2^k, not r_2^{k-1}, is the optimal solution to (SP5I), we have:

$$h_o'^k + g_o'^k \leq h_o'^{k-1} + g_o'^{k-1} \qquad (7-4)$$

Now add the fuel consumption at point r_1^{k-1} to both sides of (7-4) to get:

$$h_o'^k + g_o'^k + h_o^{k-1} + g_o^{k-1} \leq h_o'^{k-1} + g_o'^{k-1} + h_o^{k-1} + g_o^{k-1} \qquad (7-5)$$

Now, given r_2^k and the fact that r_1^k, not r_1^{k-1}, is the optimal solution to (SP5II), we have

$$g_o^k + h_o^k < g_o^{k-1} + h_o^{k-1} \qquad (7-6)$$

Now, add the fuel consumption at point r_2^k to both sides of (7-6) to get

$$g_o^k + h_o^k + g_o'^k + h_o'^k \leq g_o^{k-1} + h_o^{k-1} + g_o'^k + h_o'^k \qquad (7-7)$$

And equation (7-3) follows from (7-5) and (7-7) Q.E.D.

4) a- If $r_1^{k-1} = r_1^*$, then $r_2^k = r_2^*$

b- If $r_2^k = r_2^*$, then $r_1^k = r_1^*$

<u>Proof</u>:

This follows from Facts # 1 and 2. Q.E.D.

This means, if the procedure reaches any of the global optimal refueling points, it will immediately give us the other one and the procedure will terminate there and will not drift away from it. Thus, we have a stable solution procedure.

Thus far, we showed that the procedure will at least converge to the local optimal solution, and if it reaches the global optimal solution it will stay there. But there is evidence it might converge to the global optimal solution. A set of problems (similar to those of Table 4-2) representing a variety of possible geographic configuration and complexities were solved. All converged to the global optimal solution. Therefore, we conjecture that the procedure will converge globally for real world problem.

CHAPTER VIII

SUMMARY AND RECOMMENDATIONS FOR FUTURE RESEARCH

Summary

In this dissertation, we investigated and analyzed the general A/C mid-air refueling problem in an attempt to characterize and find the optimal solution. It was found that eventhough a mathematical model for the general problem could have been formulated, it would have provided little insight and would have been too complicated to solve. Instead, the analysis approach used in this dissertation consisted of breaking the general problem down into smaller and simpler (sub)problems that were addressed in order of increasing complexity. Work that has been done regarding this problem includes:

1. Deriving the fuel-related functions, which are summarized in Table 2-1.
2. Solution of the first problem (P1). It has been proven that the minimum number of A/C must be used and the cargo weight should be distributed equally among those A/C used.

3. Solution of the second problem (P2). S-convexity of the problem has been shown and a solution procedure has been devised that utilizes the relationship between the various decision variables at optimality. Also, a proof of the convergence of this procedure is provided. The solution of (P1) and (P2) was essential to facilitate the solution of the other problems.
4. Solutions of the third and fourth problems (P3 and P4). They are based on the solutions of (P1) and (P2) and the insight obtained from them. The main contribution here was to find the optimal weights (w_i^*'s). We can say that the general N A/C problem with one refueling has been solved optimally.
5. Solution of the fifth problem (P5). We proposed an iterative solution procedure where, at each iteration, one refueling point is being fixed while we solve for the other. Then, using the result of the first subproblem as an input to the second, we resolve for the original refueling point. Each subproblem is essentially problem (P2) with minor changes. We showed that this procedure will at least converge locally, and that the solution is stable. Moreover, numerical results of some test problems support the conjecture that this procedure will converge globally for real world problem. The extension of this procedure to the multi-refueling case is straightforward. We would fix all refueling points but one which we solve for. Then we repeat this process, solving for a different refueling point at each time until convergence is reached. A global convergence proof for this problem, however, is beyond the scope of this dissertation.

Problem (P6) and (P7) were not solved. Note that if each w_i is decided upon, then we will end up with a combination of (P1), (P2) and (P5) whose solutions are known. The big question that is left is: what are the values for the optimal weights (w_i^*'s)? As a conjecture, we believe that the solution to problem (P7) might be similar to that of (P4). To be more specific, recall the definitions of Z_o and Z_1 (Chap. V, page 62), the definition of M_{min} and Q (Chap. VI, page 77), and the definitions of ℓ_D and ℓ_R (Chap. VI, page 81). Similarly, define Z_2, Q_2 and ℓ_{R2} as follows:

Z_2 = maximum cargo weight one transport A/C can carry from "O" to "D" by refueling twice ($Z_2 > Z_1$).

Q_2 = minimum number of transport A/C needed to transport the whole cargo (W) from "O" to "D" when aerial refueling is used twice for all of them = $\lceil W/Z_2 \rceil$.

ℓ_{R2}= a certain number of A/C used which will need to refuel twice in mid-air.

We conjecture that the optimal solution will be one of the following options:

a) The (M_{min} D) option where M_{min} A/C fly directly, each with w_i^* = W/M_{min}.

b) The (QR) option where Q A/C refuel once in mid-air, each with w_i^* = W/Q.

c) The (Q_2 R_2) option where Q_2 A/C refuel twice in mid-air, each with w_i^* = W/Q_2.

d) The ($\ell_D D$ $\ell_R R$ $\ell_{R2} R_2$) option(s) where ℓ_D A/C fly directly, ℓ_R A/C refuel once and ℓ_{R2} A/C refuel twice. The optimal weight for the different A/C is:

$$w_i^* = Z_o \qquad \text{for } i = 1, 2 \ldots \ell_D$$

$$w_i^* = Z_1 \qquad \text{for } i = \ell_D + 1, \ldots \ell_D + \ell_R$$

$$w_i^* = \frac{W - \ell_D Z_o - \ell_R Z_1}{\ell_{R2}} \qquad \text{for } i = \ell_D + \ell_R + 1, \ldots \ell_D + \ell_R + \ell_{R2}$$

Future Research Areas

As noted above, several different problems are still open for further research. Following are some of them:

1. Proving the global convergence of the solution procedure of problem (P5);
2. Solving problems (P6) and (P7). The main job is to find the relationship between the different weights at optimality;
3. Extending the research to the case where there are more than one type of transport A/C and/or tanker A/C;
4. Investigating the case of a forbidding region (i.e., region where the refueling cannot occur and/or the A/C cannot fly, such as a particular country's air space).

APPENDIX A

GLOSSARY OF NOTATIONS AND DEFINITIONS

1)	N	=	number of available A/C
2)	n	=	transport A/C number where $n = 1,2,\ldots N$
3)	EW	=	transport A/C Empty Weight
4)	$MTOW$	=	Maximum Take-off Weight of a transport A/C
5)	F_{max}	=	maximum fuel capacity of a transport A/C
	H_{max}	=	maximum fuel capacity of a tanker A/C
6)	W	=	total weight of the cargo to be transported
7)	w_n	=	weight of the portion of the cargo to be loaded in the n^{th} transport A/C, (a decision variable) where: $W = \sum_{n=1}^{N} w_n$
8)	g_n	=	initial fuel of the n^{th} A/C (a decision variable)
9)	h_k	=	initial fuel of the k^{th} tanker where $k = 1,2,\ldots,K$
10)	GW_n	=	gross weight of a transport A/C, where $GW_n = EW + g_n + w_n$
	GW_{max}	=	maximum allowed gross weight of a transport A/C
11)	$g_{max}(w_n)$	=	minimum $\{F_{max}, MTOW - w_n - EW\}$
	$G_{max}(w_n)$	=	minimum $\{F_{max}, GW_{max} - w_n - EW\}$

12) MPF(GW) = distance traveled by an A/C in miles per 1,000 lbs of fuel at a given gross weight GW

13) $R_n\ (g_n, w_n)$ = Range of the n^{th} transport A/C when its initial fuel is g_n and cargo weight is w_n

14) $FC_n\ (g_n, w_n, d)$ = Fuel Consumed by the n^{th} transport when it flies a distance d, its initial fuel is g_n and cargo weight is w_n
[Note: d must be $\leq R_n(g_n, w_n)$]

15) $FN_n\ (w_n, d)$ = exact amount of Fuel Needed to fly a distance d when the cargo weighs w_n
[d must be $\leq R_n\ (F_{max}, 0)$]

16) $R_k^*\ (h_k, w_k)$ = Range of the k^{th} tanker when its initial fuel is h_k and cargo weight is w_k (usually $w_k = 0$)

17) $FC_k^*\ (h_k, w_k, d)$ = Fuel Consumed by the k^{th} tanker when it flies a distance d, its initial fuel is h_k and cargo weight is w_k
[d must be $\leq R_k^*\ (h_k, w_k)$]

18) $FN_k^*\ (w_k, d)$ = exact amount of Fuel Needed by tanker to fly a distance d when its cargo weight is w_k [d must be $\leq R_k^*\ (H_{max}, 0)$]

19) D_{OD} = shortest distance between the origin base and destination measured along the arc of the great circle connecting the two points

20) (θ_0, ϕ_0) = spherical coordinates of the origin base (latitude and longitude)

(θ_D, ϕ_D)	=	spherical coordinates of the destination
(θ_B, ϕ_B)	=	spherical coordinates of the tanker base
(θ, ϕ)	=	any point on the sphere (earth)
21) $d_1(\theta, \phi)$	=	spherical distance between the origin base and (θ, ϕ)
$d_2(\theta, \phi)$	=	spherical distance between the destination and (θ, ϕ)
$d_3(\theta, \phi)$	=	spherical distance between the tanker base and (θ, ϕ).

APPENDIX B

MEASURE OF DISTANCE

Since a suitably small portion of the earth can be considered as a plane, the shortest distance between any two points in such a region is generally approximated by Euclidean distance. However, Litwhiler and Aly [22] showed that the Euclidean assumption for even moderately large regions (800 by 1000 miles) can lead to significant error in objective value. In one problem, they showed that using the Euclidean assumption resulted in objective error of about 18 percent and a location error of over 10,000 miles. (See Table B-1.) In this case it is more appropriate to model the earth as a sphere (even though it is not a perfect sphere, but rather a spheroid).

In going from one point to another on the surface of a sphere (earth), one traverses a curve, not a straight line. The curve having the shortest distance between two points on the surface of the sphere is the great circle. The great circle is the circle in E^3 which has a center at the center of the sphere and is drawn on the surface so that its radius is equal to the radius of the sphere. (See Figure B-1.) Thus, the shortest distance between two points on a sphere is the length of the shorter great circle arc connecting them.

Three equivalent methods of calculating this distance have been reported in the literature. All the methods use the following two facts: (i) the great circle distance between two points on a sphere is directly proportional with the radius of the sphere; (ii) the distance

TABLE B-1: Error in distance when the planar assumption is used

Angle between two points (degrees)	(1) Approximate distance using planar assumption (nautical miles)	(2) Exact distance (nautical miles)	error = (2) - (1) (nautical miles)	% error $= \frac{(2) - (1)}{(2)}$
15	888.978	891.524	2.546	0.286
30	1762.733	1783.047	20.315	1.139
45	2606.303	2674.571	68.268	2.552
60	3405.242	3566.095	160.853	4.511
75	4145.871	4457.619	311.749	6.994
90	4815.504	5349.143	533.638	9.976
105	5402.678	6240.667	837.989	13.428
120	5897.335	7132.190	1234.855	17.314
135	6291.006	8023.714	1732.708	21.595
150	6576.951	8915.238	2338.288	26.228
165	6750.271	9806.762	3056.491	31.167
180	6807.999	10698.285	3890.286	36.364

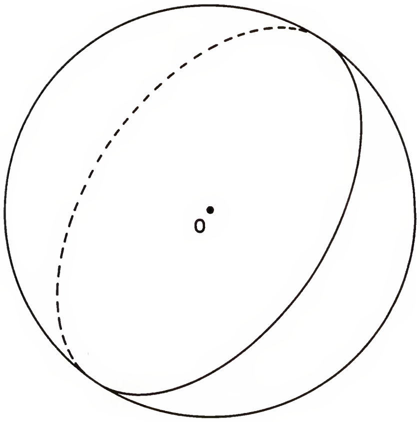


Figure B-1: The Great Circle.

between two points on a unit sphere is identical to the angle (measured in radians) between the two normals of the sphere at these points.

Therefore, distance = $R \cdot \alpha$ (B-1)

where R = the radius of the sphere and

α = the angle between the two normals at these two points (measured in Radians). (See Figure B-2.)

The three methods differ only in the way they find the angle α, but they produce the same result.

Method 1

Katz and Cooper [20] and Litwhiler and Aly [22] considered the distance as

$$d(X,Y) = 2R \arcsin \left(\frac{||X - Y||}{2R}\right) \qquad \text{(B-2)}$$

where

X and Y are two points on a sphere with radius R and centered at the origin,

d(X,Y) = the distance between X and Y measured along the surface of the sphere, and

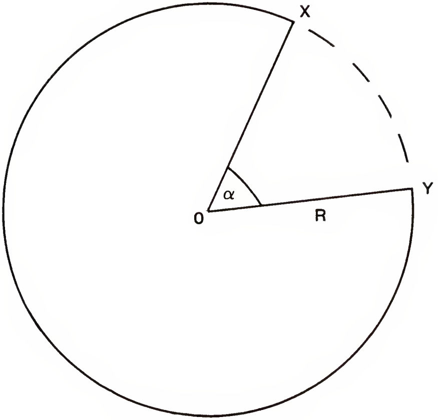
$||\cdot||$ = the Euclidean norm.

This is because

$$\operatorname{Sin}\left(\frac{\alpha}{2}\right) = \frac{\frac{||X - Y||}{2}}{R} = \frac{||X - Y||}{2R}$$

$$\frac{\alpha}{2} = \arcsin \left(\frac{||X - Y||}{2R}\right) \qquad \text{(B-3)}$$

(See Figure B-3.) And since $d(X,Y) = R\alpha$, then equation (B-2) follows.



$$d(X,Y) = R\alpha$$

Figure B-2: The spherical distance between two points.

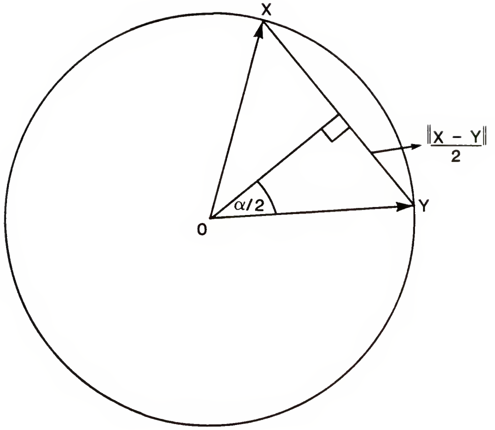


Figure B-3: Finding the angle α using method 1.

The two following methods use a third fact. If α is the angle between any two nonzero vectors X and Y, then

$$\text{Cos } \alpha = \frac{X \cdot Y}{||X|| \quad ||Y||} \tag{B-4}$$

Now, if X and Y are on a sphere, then

$$||X|| = ||Y|| = R$$

Furthermore, if we consider a unit sphere, then

$$R = 1, \text{ thus Cos } \alpha = X \cdot Y$$

$$\alpha = \arccos (X \cdot Y) \tag{B-5}$$

and since the distance = $R\alpha$, and R = 1, then α itself becomes the measure of distance.

Method 2

By representing X and Y in spherical coordinates as (R, θ_1, ϕ_1) and (R, θ_2, ϕ_2), respectively, instead of the cartesian coordinates (x_1, x_2, x_3) and (y_1, y_2, y_3), then

$$x_1 = R \sin \phi_1 \cos \theta_1$$

$$x_2 = R \sin \phi_1 \sin \theta_1$$

$$x_3 = R \cos \phi_1$$

$$y_1 = R \sin \phi_2 \cos \theta_2$$

$$y_2 = R \sin \phi_2 \sin \theta_2$$

$y_3 = R \cos \phi_2$, and

$||X|| = ||Y|| = R$ as before, so

$||X|| \cdot ||Y|| = R^2$, and

$$X \cdot Y = R^2[\sin \phi_1 \sin \phi_2 \cos \theta_1 \cos \theta_2 + \sin \phi_1 \sin \phi_2 \sin \theta_1 \sin \theta_2 + \cos \phi_1 \cos \phi_2]$$
$$= R^2[\sin \phi_1 \sin \phi_2 (\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2) + \cos \phi_1 \cos \phi_2]$$

By recognizing that

$$\cos(\theta_2 - \theta_1) = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2,$$

we arrive at

$$\cos \alpha = \sin \phi_1 \sin \phi_2 \cos(\theta_2 - \theta_1) + \cos \phi_1 \cos\phi_2 \qquad \text{(B-6)}$$

Notice that θ = the geographical longitude, but ϕ is not the geographical latitude. In fact, $\phi = \pi/2$ - latitude, and it is called the co-latitude. So, if one would rather work with longitude and latitude instead, then equation (B-6) becomes

$$\cos \alpha = \cos \phi_1' \cos \phi_2' \cos(\theta_2 - \theta_1) + \sin \phi_1' \sin \phi_2' \qquad \text{(B-7)}$$

where ϕ_i' = the latitude of point i. Equation (B-7) was used by Drezner and Wesolowsky [12] and originally reported by Donney [9]. This is the formula that is used throughout this study.

Method 3

Lex $X = (x_1, x_2, x_3)$ be a point on a unit sphere. We can represent X as $(1,\theta,\phi)$ where:

ϕ = the latitude of X

θ = the longitude of X, then

$x_1 = \cos \phi \cos \theta$

$x_2 = \cos \phi \cos \theta$

$x_3 = \sin \phi$.

Let (a,b,c) be the cartesian coordinates of a point Y on the surface of a unit sphere. Then the distance between X and Y, using equation (B-5), is:

$$d(X,Y) = \alpha = \arccos\ (X,Y)$$

$$= \text{arcos}\ (a \cos\phi \cos\theta + b \cos\phi \sin\theta + c \sin\phi) \qquad \text{(B-8)}$$

Litwhiler and Aly [22] used this formula to characterize the shortest distance between any two points on the sphere.

APPENDIX C

SPHERICAL CONVEXITY

Sets and functions defined on the surface of a sphere do not satisfy the usual convexity definition in E^3. However, this definition has been extended and modified [1,3,10,12] for the case of spherical sets and functions. Several properties and results of spherical convexity theory have been developed which are, to a certain extent, similar to the convexity theory in E^3. For a complete treatment of this subject, see Drezner and Wesolowsky [12], Aly et al., [1], Drezner [10] and Ayken [3]. Among the several definitions, results, and properties found in the references cited above, only the most important ones are included in the following list.

Definition C-1 (Drezner and Wesolowsky [12])

A set of points on a sphere is said to be spherically convex (s-convex set) if, for any two points of the set, the whole shorter great circle arc connecting them is included in the set.

Definition C-2 (Drezner and Wesolowsky [12])

A spherically convex combination of two points r_1, r_2 is defined as the point $r = \rho\ (r_1,r_2,\lambda)$ that lies on the shorter great circle arc between r_1 and r_2 such that the great circle distance between r_1 and r is $\lambda * d(r_1,r_2)$ for) $0 \leq \lambda \leq 1$.

Definition C-3 (Drezner and Wesolowsky [12])

A function $f(r)$ over an s-convex set is said to be a spherically convex function (s-convex function) if for every $0 \leq \lambda \leq 1$:

$f(\rho(r_1, r_2, \lambda)) \leq \lambda f(r_1) + (1 - \lambda) f(r_2)$ for any element r_1 and r_2 of the set.

Theorem C-1 (Drezner and Wesolowsky [12])

Points within a circle on a unit sphere with radius $\leq \pi/2$ form an s-convex set.

Theorem C-2 (Drezner and Wesolowsky [12])

The distance from a given point r is an s-convex function within a circle of radius $\pi/2$ and center r.

Theorem C-3 (Ayken [3])

Every minimum of an s-convex function over an s-convex set is a global minimum.

Theorem C-4 (Ayken [3])

If an s-convex function f(r) over an s-convex set D has two different minima, r_1, $r_2 \in D$, then any s-convex combination of them is also a global minimum.

Theorem C-5 (Ayken [3])

An s-strictly convex function over an s-convex set D possesses a unique minima.

Theorem C-6 (Aly et al. [1])

The search for an optimal solution to the spherical Weber problem, where demand points are not located entirely on a great circle arc, can be restricted to the spherical convex hull of the demand points.

Theorem C-7 (Drezner [10])

In the spherical Weber problem, if all demand points are located on a great circle, so is the optimal solution.

Theorem C-8 (Ayken [3])

In the spherical Weber problem, if all demand points are included within a spherical circle of radius $\leq \pi/4$, then every minima is a global minimum.

Theorem C-9 (Ayken [3])

For the spherical Weber problem with $m \geq 3$ and $\beta_i \geq 0$, if all demand points are included within a spherical disk or radius $\pi/2$ and at least three demand points are not spherically colinear, then the problem possesses a unique minimum.

APPENDIX D

SPECIAL CONVEXITY PROPERTIES FOR PROBLEM (P2)

Theorem D-1

The function FN (w,d) is s-convex over a spherical disc of radius $\leq \pi/4$.

Proof

Let r, r_1, and r_2 be any three points in the described spherical disc. Let $r_3 = \rho(r_1,r_2,\lambda)$ for some λ ($0 \leq \lambda \leq 1$) (See definition C-2) Also let $d(r_i)$ = the s-distance between r and r_i for i = 1, 2, 3. Thus $d(r_i) \leq \pi/2$ because the radius of the spherical disc is $\leq \pi/4$. Therefore $d(r_i)$ is a s-convex function $\rightarrow d[\rho(r_1,r_2,\lambda)] \leq \lambda\, d(r_1) + (1 - \lambda)\, d(r_2)$ or equivalently $d(r_3) \leq \lambda\, d(r_1) + (1 - \lambda)\, d(r_2)$. Since FN (w,d) is a convex and increasing function of d, then applying FN to both sides of the above inequality results in

$$FN\,[w,d(r_3)] \leq FN\,[w, \lambda d(r_1) + (1 - \lambda)\, d(r_2)]$$

$$\leq \lambda\, FN\,[w,d(r_1)] + (1 - \lambda)\, FN\,[w,d(r_2)]$$

Thus FN [w,d] is s-convex Q.E.D.

The following theorem is an extension of theorem C-6 of Aly et al. [1] to the case of problem (P2).

Theorem D-2

The search for the optimal refueling point of problem (P2) can be restricted to the spherically convex hull of the three base points (the origin, the destination, and the tanker base).

Proof

Let V be the spherically convex hull of the three points and X be a point on the sphere such that $X \notin V$. Let d_1, d_2 and d_3 be the spherical distance between X and the origin, the destination, and the tanker base, respectively.

Aly et al. [1] proved that there exists a point $X' \varepsilon V$ that dominates X in the spherical distance with respect to the three base points. That is, if d_1', d_2', and d_3' are the distance between X' and the three points, then $d_i' \leq d_i$ for i = 1, 2, 3.

Since fuel consumption increases with distance, (i.e., $FN(\cdot,d_2)$, $FC(\cdot,\cdot,d_1)$, $FN^*(\cdot,d_3)$ and $FC^*(\cdot,\cdot,d_3)$ are all increasing functions of the distance), then

the objective function value at X'

$\leq$ the objective function value at X.

Q.E.D.

The following lemma will be used later to arrive at a sufficient condition which guarantees s-convexity of the objective function of problem (P2).

Lemma (D-1)

In order to prove s-convexity of the objective function of problem (P2), it is sufficient to show that at any feasible point (after transferring the North Pole to it) the second derivative with respect to ϕ for any given θ (direction) is non-negative.

Proof

The proof will be given in three (3) parts: Part (a) gives a geometrical interpretation of convexity in Euclidean Space; Part (b) shows how to apply this using polar coordinates; Part (c) proves the above Lemma using analogy.

(a) The definition of (regular) convexity in Euclidean Space when cartesian coordinates are used states that: A function $f(x)$ is convex at a point $\bar{x}$ of a convex set S if

$$f[\lambda\bar{x} + (1 - \lambda)\, x\,] \leqslant \lambda\, f(\bar{x}) + (1 - \lambda)\, f(x)$$

for each $\lambda \varepsilon\ (0,1)$ and for each $x \varepsilon S$, $x \neq \bar{x}$ (see Reference [5], page 109). Note that $\lambda\bar{x} + (1 - \lambda)\, x$ is any point on the line segment joining $\bar{x}$ and x. (This is true because we are using cartesian coordinates, but not true if we are using any other coordinate system.) This means f has to be convex along any straight line passing through $\bar{x}$, no matter what its direction is.

(b) Thus, proving the convexity of a function at a point would be easy if we could implement this geometric insight.

Now, see what happens when we transfer the origin to the point under consideration. Using polar coordinates (θ,r), a certain line which passes through the (new) origin is defined by θ = constant = C, while r is allowed to vary. Along this line (or direction), the function is, in essence, a function of one variable, namely r. Thus the function is convex at the origin and along that particular line if we have

$$\left.\frac{\partial^2 f(c,r)}{\partial r^2}\right|_{r=0} \geqslant 0$$

Moreover the function is convex at the origin if

$$\left.\frac{\partial^2 f(\theta,r)}{\partial r^2}\right|_{r=0} \geq 0 \qquad \text{for all } \theta$$

(c) Now moving back to the case where the function is spherical, we can prove s-convexity in a similar fashion.

First, note that the great circle distance is the shortest distance between any two points on the surface of the sphere, and it is analagous to the straight line distance in E^3.

Second, recall the following two definitions (which can be found, among others, in Appendix C):

Def. C-2 (Drezner and Wesolowsky [12])

A spherically convex combination of two points r_1, r_2 is defined as the point $r=\rho\ (r_1,r_2,\lambda)$ that lies on the shorter great circle arc between r_1 and r_2 such that the great circle distance between r_1 and r is $\lambda * d\ (r_1,r_2)$ for $\lambda \varepsilon\ (0,1)$.

Def. C-3 (Drezner and Wesolowsky [12])

A function f(r) over an s-convex set is said to be a spherically convex function (s-convex function) if for every $0 \leq \lambda \leq 1$:

$$f[\rho(r_1,r_2,\lambda)] \leq \lambda\ f(r_1) + (1 - \lambda)\ f(r_2)$$

for every element r_1 and r_2 of the set.

This implies that the function must be convex along any great circle track. Thus, in order to prove that the function is s-convex at a point, we first transfer the North Pole (NP) to that point. Note that along any great circle arc that passes through the (new) NP, we have θ =

constant = C while ϕ varies. Thus f becomes a function of one variable, namely ϕ [see Figure (D-1)].

So f is s-convex at NP along that particular great circle track if:

$$\left.\frac{\partial^2 f(c, \phi)}{\partial \phi^2}\right|_{\phi=0} \geqslant 0$$

And in order for the function to be s-convex at NP we must have

$$\left.\frac{\partial^2 f(\theta, \phi)}{\partial \phi^2}\right|_{\phi=0} \geqslant 0 \qquad \text{for all } \theta \qquad \text{Q.E.D.}$$

<u>Condition for the s-convexity of the objective function of problem (P2)</u>

We saw from Lemma (D-1) that the objective function is s-convex if at any point (after transferring the North Pole to it) the second derivative with respect to ϕ for any given θ (direction) is non-negative. Therefore, to prove the s-convexity of the objective function, we need to show that this is the case at all feasible points or give a condition(s) under which it is true. However, a general formula for the second derivative is mathematically untractable and does not allow us to draw any helpful conclusion. Therefore, we will adopt another approach.

We want to construct (manufacture) a worst-case situation where the second derivative at some point would attain its minimum value it could ever achieve. If it is non-negative or if we find a condition under which it is non-negative, then we are done, and the problem is s-convex. This is true since, by construction, all other cases will produce larger values of the second derivative.

This can be doen by positioning the tanker base ("B"), positioning the refueling point and by selecting the direction we take in such a way that minimizes the second derivative. This selection is based on some characteristics inferred when looking at the different terms of which the second derivative is composed.

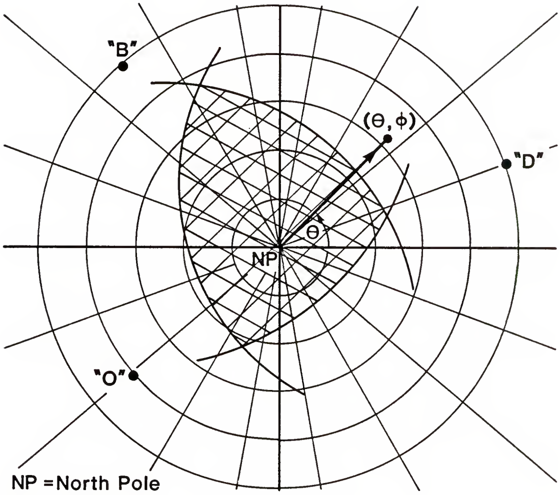


Figure D-1: Transferring the North Pole to a point under consideration.

The first thing we found was that the value of the second derivative is reduced considerably if the refueling point is positioned over the tanker base ("B"). Moreover, if "B" is within a certain distance (for the C-5A A/C this distance is at least 3,950 nautical miles) from the destination "D", then the second derivative is minimized if "B" itself is located on the great circle arc joing the origin ("O") and the destination "D". Note that the position of the minimum point simplifies the expression for the second derivative and makes it easy to draw some conclusions. In fact, the second derivative is found to be non-negative. Since this is the worst possible case, given that "B" is within a certain distance from "D," then the function is s-convex. This is exactly the condition given in Theorem (D-3). However, it is not easy to prove s-convexity if the feasible points are greater than 3,950 nautical miles from "D" because then we get a much more complicated expression for the second derivative. That is because the minimum point no longer lies on the great circle route. In that case, the expression for the second derivative is no longer simple enough to draw similar conclusions from. It should be noted, however, that 3,900 nautical miles is roughly the maximum range for a fully loaded C-5A.

In summary, when the feasible points are within a certain distance from the destination ("D"), the minimum value of the second derivative occurs if "B" is located on the great circle joing "O" and "D," and if the refueling point itself is positioned on top of "B." That is why the condition for convexity seems to be independent of the distance from the refueling points to either "B" or "O."

Now, we state this condition as theorem (D-3) with an example and a proof.

Theorem (D-3)

The objective function of problem (P2) is s-convex at any feasible point that satisfies the following condition:

$$\frac{a^2}{-a_1 R} \geq (\tan \phi_D + 2\ \phi_D) \qquad \text{(D-1)}$$

where

ϕ_D = the distance (measured in radians) between that point and the destination ("D") and

$a = a_o + a_1\ (EW + w_o)$, where

EW = A/C Empty Weight

w_o = cargo weight

R = Radius of earth, and

a_o and a_1 are the parameters of the MPF function

(see equation (2-1) and Table 2-1)

That is, if the points are within a specific distance ($\bar{\phi}_D$) from the destination then the function is s-convex at those points, where $\bar{\phi}_D$ is the solution to the following equation:

$$\frac{a^2}{-a_1 R} = \tan \bar{\phi}_D + 2\ \bar{\phi}_D \qquad \text{(D-2)}$$

and $\phi_D \leq \bar{\phi}_D$

Note that $\bar{\phi}_D$ depends on the cargo weight as well as the type of A/C used.

In Table (D-1), we present values of $\bar{\phi}_D$ which correspond to different cargo weights (w_o) for the C-5A A/C.

TABLE D-1: The distance $\bar{\phi}_D$ as a function of cargo weight (w_o) for C5-A

w_o (lbs)	$\bar{\phi}_D$ measured in		
	nautical miles	degrees	radians
0	4,594	77.3	1.3497
50 k	4,481	75.4	1.3165
100 k	4,338	73.0	1.2746
150 k	4,160	70.0	1.2222
200* k	3,950	66.5	1.1611

*200 k lbs. is the maximum cargo weight for the C-5A A/C due to structural weight limitations.

Note that the maximum value of ϕ_D we would hope for is $\pi/2$ (or 90°). (See theorems (C-1) and (C-2) of Appendix C.) Before we derive condition (D-1), we would like to give an example that will show the implication of this condition.

Example

Suppose the Origin ("O") and Destination ("D") are 90° (or $\pi/2$) apart. Moreover, the cargo weighs 200 k lbs. We would like to know whether or not the function is s-convex over all the feasible region.

Solution

Since the two bases are 90° apart then

D_{OD} = 6,220 status miles = 5349.12 nautical miles

w_o = 200 k lbs

The data for the C-5A are as follows:

$a_o = 36.283$, $a_1 = -0.027$

MTOW = 670 k lbs., GW_{max} = 720 k lbs.

EW = 320 k lbs , F_{max} = 300 k lbs of fuel

$$\vec{g}_{max}(w_o) = \text{Min}\left\{F_{max},\ \text{MTOW} - \text{EW} - w_o\right\}$$

$$= 150 \text{ k lbs of fuel}$$

$R_1 = R(w_o, g_{max}(w_o))$ = 3,032 nautical miles (or 51°)

$$G_{max}(w_o) = \text{Min}\left\{F_{max},\ GW_{max} - \text{EW} - w_o\right\}$$

$$= 200 \text{ k lbs of fuel}$$

$R_2 = R(w_o, G_{max}(w_o))$ = 3,900 nautical miles (or 65.7°)

But for the C-5A with 200 k lbs. of cargo, we have $\bar{\phi}_D = 3950$ nautical miles = 66.5°, and since R2 (in degrees) = 65.7° < $\bar{\phi}_D$, then the function is s-convex over all the feasible region. (See Figure D-2.) This implies that for any A/C we can have a s-convex function even though the distance between the origin and destination is as large as $\pi/2$ (or 90°) provided that $\phi_D \leq \bar{\phi}_D$. This shows that what is important is the distance from the feasible points to the destination, not the distance from the origin to the destination (although the latter must be $\leq \pi/2$ in all cases). In fact, for the C-5A, we always will have a s-convex function over the feasible region regardless of the cargo weight, provided that it is not more than 200 k lbs. and the distance between the origin and the destination is not more than $\pi/2$. (This is derived from the previous table and the corresponding range for the different weights.)

Proof

In this proof, we will derive condition (D-1) using the worst-case method. We will construct a situation where we will be able to have the minimum value ever possible of the second derivative. Then, if need be, we impose any condition to guarantee its non-negativity.

How is this done? First, we study the expression for the second derivative and infer any desired characteristics (or requirements) needed in: (1) the position of "B"; (2) the position of the refueling points and (3) the direction of the derivative (or θ) in order to make the second derivative as small as possible. Second, we will look for the points and the direction which meet these requirements. Finally, we ascertain whether the derivative is non-negative. If not, we find out what makes it non-negative.

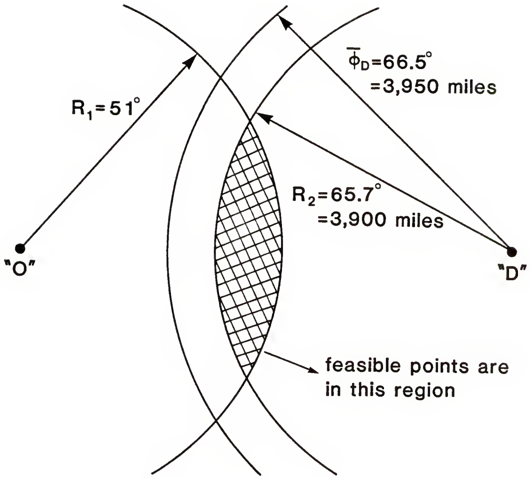


Figure D-2: An example shows the implication of Theorem D-3 for the C5-A.

Let $V(\theta,\phi) = \sum_{i=1}^{4} fi\ (\theta,\phi)$, then

$\frac{\partial V(\theta,\phi)}{\partial\ \phi}\Big|_{\phi=0}$ = derivative of $V(\theta,\phi)$ with respect to ϕ (evaluated at $\phi = 0$) in the direction (represented by the angle θ) that we take

i.e., the derivative at the north pole (where $\phi = 0$) depends on the angle θ.

$\frac{\partial^2 V(\theta,\phi)}{\partial\phi^2}\Big|_{\phi=0}$ = the second derivative,

and it also depends on θ.

$$\frac{\partial^2 V(\theta,\phi)}{\partial\phi^2}\Big|_{\phi=0} = \frac{R}{\sqrt{E_1}}\left\{\cot\ \phi_O\ \sin^2(\theta-\theta_O) + \cos^2\ (\theta-\theta_O)\ \frac{a_1 R}{E_1}\right\}$$

$$+ \frac{R}{\sqrt{E_2}}\left\{\cot\ \phi_D\ \sin^2\ (\theta-\theta_D) - \cos^2\ (\theta-\theta_D)\ \frac{a_1 R}{E_2}\right\}$$

$$+ \frac{R}{\sqrt{E_3}}\left\{\cot\ \phi_B\ \sin^2\ (\theta-\theta_B) + \cos^2\ (\theta-\theta_B)\ \frac{a_1 R}{E_3}\right\}$$

$$+ \frac{R}{\sqrt{E_4}}\left\{\cot\ \phi_B\ \sin\ (\theta-\theta_B) - \cos^2\ (\theta-\theta_B)\ \frac{a_1 R}{E_4}\right\} \qquad \text{(D-3)}$$

where $a_1 < 0$ and

$$E_1 = (a + a_1\ g_o)^2 - 2\ a_1\ R\phi_O \qquad \text{(D-4)}$$

$$E_2 = a^2 + 2a_1\ R\phi_D \qquad \text{(D-5)}$$

$$E_3 = (a_o' + a_1\ h_o)^2 - 2a_1 R\phi_B \qquad \text{(D-6)}$$

$$E_4 = (a_o')^2 + 2a_1 R\phi_B \qquad \text{(D-7)}$$

Note that, in this case, the variables are the θ_i's and ϕ_i's, depending upon the location of the point, together with the direction θ we take. (See Fig. (D-3).)

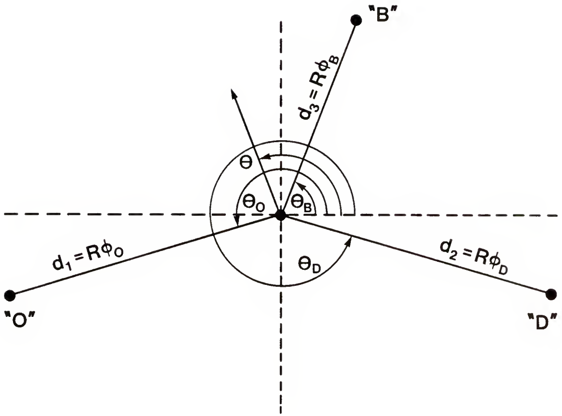


Figure D-3: The values of θ, θ_i's and ϕ_i's for a point after transferring the North Pole to it.

Also, note that all the terms of the second derivative are greater than or equal to zero except the following two:

$$\frac{R}{\sqrt{E_1}}\left[\cos^2(\theta - \theta_0)\,\frac{a_1 R}{E_1}\right] = \frac{a_1 R^2}{E_1^{3/2}}\cos^2(\theta - \theta_0), \text{ and}$$

$$\frac{R}{\sqrt{E_3}}\left[\cos^2(\theta - \theta_B)\,\frac{a_1 R}{E_3}\right] = \frac{a_1 R^2}{E_3^{3/2}}\cos^2(\theta - \theta_B)$$

because $a_1 < 0$.

It is now easy to find the desired point and direction by noticing that the following four conditions are desired in the points and direction we are looking for: [See equation (D-3).]

1. have $\cos^2(\theta - \theta_0) \to 1$

2. have $\cos^2(\theta - \theta_3) \to 1$

3. have $\phi_B \to 0$ to make $E_3\downarrow$ and $E_4\uparrow$ (See (D-6) and (D-7).)

4. have $\cos^2(\theta - \theta_D) \to \begin{cases} 1 \text{ if } \cot\phi_D \geq \dfrac{-a_1 R}{E_2} \\ 0 \text{ if } \cot\phi_D \leq \dfrac{-a_1 R}{E_2} \end{cases}$

We will show, in incremental fashion, that there exist points and a direction that have all these characteristics. The first two characteristics are satisfied by a point along the great circle track joining "0" and "B" and by having θ located along that track too (see Figure D-4). That is $\theta_0 = \theta_B + 180^0$ and $\theta = \theta_B$ or $\theta = \theta_0$.

Moreover, the third characteristic will also be satisfied as well if we let $\phi_B = 0$ i.e., choose the north pole to be at "B." (See Figure D-5.) Thus, we have

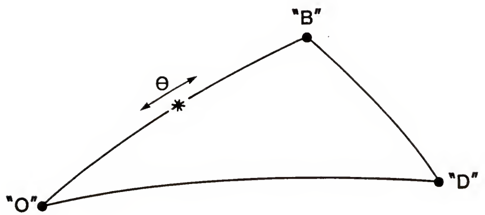


Figure D-4: A point along the Great Circle track joining "O" and "B" have the first two desired characteristics.

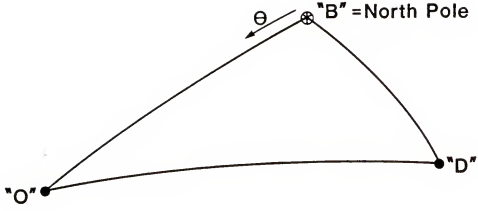


Figure D-5: Point "B" (North Pole) has the first three desired characteristics.

$$\frac{\partial^2 V(\theta,\phi)}{\partial \phi^2}\bigg|_{\phi=0} = \frac{a_1 R}{E_1^{3/2}} + \frac{R}{\sqrt{E_2}} \{\cot \phi_D \sin^2(\theta - \theta_D) - \cos^2(\theta - \theta_D)\frac{a_1 R}{E_2}\}$$

$$= a_1 R^2 \{\frac{1}{E_3^{3/2}} - \frac{1}{E_4^{3/2}}\}$$

$$= \frac{a_1 R}{E_1^{3/2}} + \frac{R}{\sqrt{E_2}} \{\cot \phi_D \sin^2(\theta - \theta_D) - \cos^2(\theta - \theta_D)\frac{a_1 R}{E_2}\}$$

$$+ a_1 R \{\frac{1}{(a_o' + a_1 h_o)^3} - \frac{1}{a_o'^3}\}$$

Note that the condition $\cot \phi_D \geq \frac{-a_1 R}{E_2}$ is equivalent to $\frac{a^2}{-a_1 R} \geq$ $(\tan \phi_D + 2\phi_D)$. This represents a region around "D." See Figure D-6.

For the region where $\cot \phi_D \geq \frac{-a_1 R}{E_2}$, we will have $\cos^2(\theta - \theta_D) = 1$ if "B" (or the NP) happens to be on the line joining "O" and "D" (see Figure D-7). In other words, $\theta_O = \theta_D + 180^0$ and $\theta = \theta_O$ or $\theta = \theta_D$, so

$$\frac{\partial^2 V(\theta,\phi)}{\partial^2 \phi^2}\bigg|_{\phi=0} = \frac{a_1 R^2}{E_1^{3/2}} - \frac{a_1 R^2}{E_2^{3/2}} + a_1 R^2 \{\frac{1}{(a_o' + a_1 h_o)^3} - \frac{1}{a_o'^3}\}$$

For this case, the minimum is achieved by setting the following quantities as follows:

$d_1 = R(w_o, g_o)$ where

$$g_o = \begin{cases} g_{max}(w_o) & \text{if } R(w_o, g_{max}) \leq D_{OD}/2 \\ \dfrac{-a}{a_1} + \dfrac{\sqrt{a^2 + 2a_1(D_{OD}/2)}}{a_1}, & \text{otherwise (i.e., } g_o \text{ is just enough to take us half way).} \end{cases}$$

Of course, $d_2 = D_{OD} - d_1$.

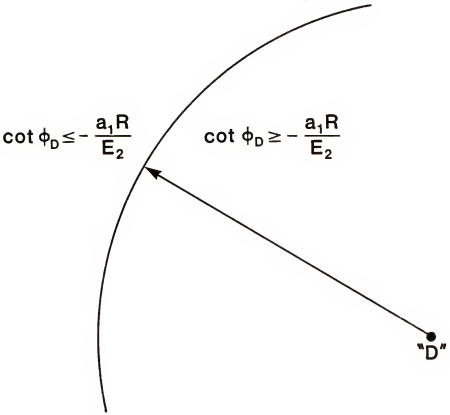


Figure D-6: A region around "D" where cot $\phi_D > -\frac{a_1 R}{E_2}$.

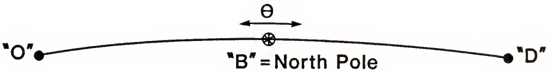


Figure D-7: Point "B" (North Pole) will have the four desired characteristics if it is located along the line joining "O" and "D."

This means $h_o = f_2$; that is, supply the transport A/C with all the fuel it needs for the second leg.

$$\left.\frac{\partial^2 V(\theta,\phi)}{\partial\phi^2}\right|_{\phi=0} = - a_1 R^2 \left\{\frac{1}{(a^2 + 2a_1 d_2)^{3/2}} - \frac{1}{a^3}\right\}$$

$$+ a_1 R^2 \left\{\frac{1}{(a'_o + a_1 h_o)^3} - \frac{1}{a'^3_o}\right\}$$

Now it remains to show that the above expression is always ≥ 0. Note that $h_o = f_2$. So

$$h_o = \frac{-a}{a_1} + \frac{\sqrt{a^2 + 2a_1 d_2}}{a_1}$$

. Writing it differently, yields

$(a + a_1 h_o)^2 = a^2 + 2a_1 d_2$. Therefore,

$$\left.\frac{\partial^2 V(\theta,\phi)}{\partial\phi^2}\right|_{\phi=0} = - a_1 R^2 \left\{\frac{1}{(a + a_1 h_o)^3} - \frac{1}{a^3}\right\}$$

$$+ a_1 R^2 \left\{\frac{1}{(a'_o + a_1 h_o)^3} - \frac{1}{a'^3_o}\right\}$$

$$= -a_1 R^2 \left\{\frac{a^3 - (a + a_1 h_o)^3}{a^3 (a + a_1 h_o)^3}\right\} + a_1 R \left\{\frac{a'^3_o - (a'_o + a_1 h_o)^3}{a'^3_o \, (a'_o + a_1 h_o)^3}\right\}$$

$$\left.\frac{\partial^2 V(\theta,\phi)}{\partial\phi^2}\right|_{\phi=0} = -a_1 R \, (-a_1 h_o) \left[\frac{3a(a + a_1 h_o) + a_1^2 h_o^2}{a^3 (a + a_1 h_o)^3}\right]$$

$$+ a_1 R \, (-a_1 h_o) \left[\frac{3a'_o(a'_o + a_1 h_o) + a_1^2 h_o^2}{a'^3_o (a'_o + a_1 h_o)^3}\right] \qquad \text{(D-8)}$$

Now, since $a_1 < 0$ and $a = a'_o + a_1 w_o$, then $a < a'_o$ and $a + a_1 h_o < a'_o + a_1 h_o$ $\rightarrow$

$$\frac{3}{a^2(a + a_1h_o)^2} > \frac{3}{a_o'^2(a_o' + a_1h_o)^2}$$

which is equivalent to

$$\frac{3a(a + a_1h_o)}{a^3(a + a_1h_o)^3} > \frac{3a_o(a_o' + a_1h_o)}{a_o'^3(a_o' + a_1h_o)^3} \tag{D-9}$$

and since

$$\frac{a_1^2\ h_o^2}{a^3(a + a_1h_o)^3} > \frac{a_1^2\ h_o^2}{a^3\ (a_o' + h_o)^3} \tag{D-10}$$

then adding (D-9) and (D-10) yields

$$\left[\frac{3a(a + a_1h_o) + a_1^2\ h_o^2}{a^3(a + a_1h_o)^3}\right] > \left[\frac{3a_o'(a_o' + a_1h_o) + a_1^2h_o^2}{a_o'^3\ (a_o' + a_1h_o)^3}\right] \tag{D-11}$$

from which it follows after looking at equation (D-8) that,

$$\left.\frac{\partial^2 V(\theta,\phi)}{\partial\phi^2}\right|_{\phi=0} > 0 \qquad \text{Q.E.D.}$$

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BIOGRAPHICAL SKETCH

Mr. Abdulrahman Yamani was born on February 9, 1955, in Medina, Saudi Arabia. He grew up there and received his secondary school certificate (high school diploma) in 1972. He then moved to Dhahran, Saudi Arabia, where he enrolled in the University of Petroleum and Minerals (UPM). There he joined the Systems Engineering Department (SE).

During the summer of 1976 and 1977, he worked as a summer student for ARAMCO in the Computer Applications Department and the Industrial Engineering Department, respectively. He graduated with honors in June 1977 and joined the faculty of UPM as a Graduate Assistant in the SE Department, where he taught computer programming (Fortran) and Introduction to Engineering. At that time, the graduate program in the SE Department was not yet established. Therefore, in August 1978, he received a scholarship to pursue his Ph.D. degree in the United States.

He enrolled in the Industrial Engineering Department of Stanford University, where he received his MSIE in May 1980. He left Stanford in January 1981 for the University of Florida, where he joined the Industrial and System Engineering Department.

He expects to finish his Ph.D. requirements in August 1986. Officially, he will graduate in December 1986. He will go back to UPM to join the faculty of the SE department, where he will teach and conduct research. Mr. Yamani's main research interests are analysis of production and inventory systems, facility location, and optimization.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Thom J. Hodgson, Chairman
Professor of Industrial and Systems Engineering

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Louis Martin-Vega, Cochairman
Associate Professor of Industrial and Systems Engineering

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Boghos Sivazlian
Professor of Industrial and Systems Engineering

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This dissertation was submitted to the Graduate Faculty of the College of Engineering and to the Graduate School and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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